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LONG-TIME EVOLUTION DESCRIBED BY THE UNITARY GROUP OF THE MEHLER OPERATOR

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In this paper, the long-time asymptotics of the solution to the Cauchy problem is described by means of the evolution unitary group of the self-adjoint Mehler operator. Spectral analysis of the latter operator is also discussed. Bibliography: 12 *titles.*

1. Introduction

The Mehler operator[1] is a self-adjoint bounded integral operator defined by the expression

$$[Du](x) := \frac{1}{\pi} \int_0^1 \frac{u(y)}{x+y}\, \mathrm{d}y \tag{1}$$

in the space $L_2(0,1)$. This operator arises naturally when studying spectral problems for Laplacians with singular potentials supported on conical or wedge-shaped surfaces [1–3]. The corresponding problems are reduced to some functional-difference equations, and then to the perturbed Mehler operator, which is considered as a compact perturbation of the Mehler operator (1). It is remarkable that the latter model can be called explicitly solvable; this means, in particular, that its spectrum and the corresponding eigenvalues can be found explicitly. Having described the spectral properties of this operator, we then consider the asymptotic behavior of the solution to the Cauchy problem at large times ($t \to \infty$),

$$\mathrm{i}\, \frac{\partial \phi(x,t)}{\partial t} + [D\phi](x,t) = 0, \quad \phi(x,0) = f(x), \tag{2}$$

$f \in L_2(0,1)$, whose solution has the form $\phi(x,t) = \exp\{-\mathrm{i}t\, D\}\, f(x)$.

In the following sections, we use the modified Mehler–Fock transform, which diagonizes the Mehler operator, describe its spectrum and the corresponding "eigenfunctions." We also obtain the resolvent of the operator. Then this information is used to construct the evolution operator $\exp\{-\mathrm{i}t\, D\}$ corresponding to the problem (2). Applying traditional asymptotic methods to the integral representation of the solution to the Cauchy problem, we obtain estimates of its behavior at large times.

The Mehler operator can be formally studied in a similar way as the Carleman [12] or Hankel [8] operators; however, since it has some special kernel, it is reasonable and instructive to study its spectral properties directly without reference to known results for the Carleman or Hankel operators.

2. Modified Mehler–Fock transform and its application for the Mehler operator in $L_2(0,1)$

In this section, we use the known results on the traditional Mehler–Fock transform [7]. We begin by considering functions F such that $\int_0^1 \frac{|F(y)|}{\sqrt{y}} \log(1+1/y)\, \mathrm{d}y < \infty$, $F \in L_2(0,1)$. Let us

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[1]Mehler appears to be the first to diagonalize an operator with kernel $1/(x+y)$ on the semi-axis. However, later Dixon (see [10, Sec. 11.18]) solved (apparently, independently) an integral equation with this kernel on the interval $[0,1]$ by reducing it to an integral equation of convolution type on the semi-axis.

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introduce

$$\mathcal{P}_p(x) = \frac{\sqrt{p\tanh(\pi p)}}{x} P_{\mathrm{i}p-1/2}(1/x) \tag{3}$$

with the asymptotics (see [6, 8.772(1)])

$$\mathcal{P}_p(x) = \frac{\sqrt{p\tanh(\pi p)}}{x} \left(\frac{\Gamma(-\mathrm{i}p)}{\Gamma(-\mathrm{i}p+1/2)} \left[\frac{x}{2}\right]^{1/2-\mathrm{i}p} + \frac{\Gamma(\mathrm{i}p)}{\Gamma(\mathrm{i}p+1/2)} \left[\frac{x}{2}\right]^{1/2+\mathrm{i}p} \right) \left(\frac{1}{\sqrt{\pi}} + O(x^2) \right), \tag{4}$$

$x \to 0+$, $p > 0$, and $\mathcal{P}_p(x) = O(1)$ as $p \to \infty$, $1 \geq x > 0$. The functions (3) are real-valued for $p \geq 0$, in particular, $\mathcal{P}_0(x) > 0$. We recall that the Legendre function (with $x^{-1} = \cosh\alpha$) is defined in [6, 8.715]:

$$P_{\mathrm{i}\tau-1/2}(\cosh\alpha) = \frac{\sqrt{2}}{\pi} \int_0^\alpha \frac{\cos(\tau t)\,\mathrm{d}t}{\sqrt{\cosh\alpha - \cosh t}}. \tag{5}$$

The traditional Mehler–Fock transform [7, Chap. 7] leads to the required modified version of the Mehler–Fock transform (mMF transform)

$$F(x) = \int_0^\infty \mathcal{P}_p(x) F^*(p)\mathrm{d}p, \tag{6}$$

$$F^*(p) = \int_0^1 \mathcal{P}_p(x) F(x)\mathrm{d}x, \tag{7}$$

where F^* is regular at $p = 0$, absolutely integrable on $[0, \infty)$ with a locally summable derivative.

The expressions (7) and (6) (together with (3)) are considered as a modified Mehler–Fock transform. Parseval's equality takes the form [9]

$$\int_0^1 Q(x)F(x)\mathrm{d}x = \int_0^\infty Q^*(p)F^*(p)\mathrm{d}p.$$

The following relation holds:

$$\int_0^1 [F(x)]^2\mathrm{d}x = \int_0^\infty [F^*(p)]^2\mathrm{d}p.$$

We use the known Mehler formula, [7, (7-4-15)],

$$\int_1^\infty \frac{P_{\mathrm{i}p-1/2}(s)}{s+v}\mathrm{d}s = \pi \frac{P_{\mathrm{i}p-1/2}(v)}{\cosh(\pi p)}, \tag{8}$$

for $v \geq 1$ and $p \in [0, \infty)$, and the inversion formula, [7, (7-6-28)],

$$\int_0^\infty p\tanh(\pi p) \frac{P_{\mathrm{i}p-1/2}(s) P_{\mathrm{i}p-1/2}(v)}{\cosh(\pi p)}\mathrm{d}p = \frac{1}{\pi(s+v)}, \quad s, v \geq 1. \tag{9}$$

Let us replace the integration variable with $s = 1/y$ and introduce $x = 1/v$ into (8). Thus, we obtain

$$\frac{1}{\pi} \int_0^1 \frac{\mathcal{P}_p(y)}{x+y}\,\mathrm{d}y = \frac{\mathcal{P}_p(x)}{\cosh(\pi p)}. \tag{10}$$

Relation (9) implies the spectral equation for the Mehler operator D,

$$[D\mathcal{P}_p](x) = \mu(p)\mathcal{P}_p(x),$$

where

$$\mu = \mu(p) = \frac{1}{\cosh(\pi p)}. \tag{11}$$

Equation (11) describes a single-valued map of the quasimomentum $p \geq 0$ and the energy μ, $(0 < \mu \leq 1)$, so that its inversion has the form

$$p = p(\mu) = \log([1 + \sqrt{1-\mu^2}]/\mu) \geq 0.$$

We can see that $\mathcal{P}_p(x)$ is a generalized eigenfunction of the continuous spectrum $\sigma(D) = [0,1]$ of the self-adjoint operator D.[2] The generalized orthogonality and completeness of these functions take the form

$$\int_0^1 \mathcal{P}_p(x)\mathcal{P}_q(x)\mathrm{d}x = \delta(p-q), \qquad \int_0^\infty \mathcal{P}_p(x)\mathcal{P}_p(y)\mathrm{d}p = \delta(x-y).$$

3. Resolvent of the Mehler operator

We find the resolvent by solving the equation $[Du](x) - \mu u(x) = f(x)$ in $L_2(0,1)$ and, taking into account (11), we obtain $\left(\frac{1}{\cosh(\pi p)} - \mu\right) u^*(p) = f^*(p)$ and

$$u^*(p) = f^*(p)\frac{1}{\frac{1}{\cosh(\pi p)} - \mu} = f^*(p)\left(-\frac{1}{\mu} - \frac{1}{\mu^2}\frac{1}{\cosh(\pi p) - \mu^{-1}}\right).$$

We use (6) $(\mu \notin \sigma(D))$. We have

$$u(x) = -\frac{1}{\mu}\left\{f(x) + \frac{1}{\pi}\int_0^1 a(x,y;\mu)f(y)\mathrm{d}y\right\}$$

with

$$a(x,y;\mu) = \pi\int_0^\infty \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\mu\cosh(\pi p) - 1}\mathrm{d}p. \tag{12}$$

Thus, we arrive at

$$u(x) = [D-\mu]^{-1}f(x) = -\frac{1}{\mu}\{I + A_\mu\}f(x), \tag{13}$$

and A_μ is the operator defined in $L_2(0,1)$ by the expression

$$[A_\mu f](x) = \frac{1}{\pi}\int_0^1 a(x,y;\mu)f(y)\mathrm{d}y, \tag{14}$$

[2]Note that formula (9) can also be written as

$$\frac{1}{\pi(x+y)} = \int_0^\infty \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\cosh(\pi p)}\,\mathrm{d}p.$$

where the kernel is given by formula (12). It is useful to note that the kernel $a(x,y;\mu)$ solves the integral equation (Hilbert's identity for the resolvent):

$$\mu a(x,y;\mu) = \frac{1}{x+y} + \frac{1}{\pi}\int_0^1 \frac{a(y,z;\mu)}{z+x}\,\mathrm{d}z.$$

The resolvent is a holomorphic operator function $\mu \notin \sigma(D)$. It has finite limits on the sides of the cut along the segment $(0,1)$. It is also bounded at $\mu = 1$ and the kernel a admits the estimate

$$|a(x,y;\mu)| \le C\frac{1+|\log x \log y|}{\sqrt{xy}}, \quad (x,y)\in(0,1]\times(0,1]$$

in some neighborhood of $\mu = 1$.

4. Asymptotics of the solution to the Cauchy problem at large times

The solution to the Cauchy problem (2) is represented as

$$\phi(x,t) = \exp\{-\mathrm{i}tD\}f(x) = \int_0^\infty \mathrm{e}^{-\mathrm{i}tD}\mathcal{P}_p(x)f^*(p)\,\mathrm{d}p = \int_0^\infty \mathrm{e}^{-\mathrm{i}t\mu(p)}\mathcal{P}_p(x)f^*(p)\,\mathrm{d}p,$$

where $\mu(p) = 1/\cosh(\pi p)$,

$$f^*(p) = \int_0^1 \mathcal{P}_p(x)f(x)\,\mathrm{d}x$$

is assumed to be sufficiently smooth and decreasing at infinity. We use the representation (5) of the Legendre function and change the order of integration, which is justified. We have

$$\phi(x,t) = \frac{1}{\pi\sqrt{2x}}\int_0^{\alpha(x)} \frac{\mathrm{d}v}{\sqrt{\cosh\alpha(x)-\cosh v}} \times \left(\int_{-\infty}^{\infty} \mathrm{d}p\sqrt{p\tanh(\pi p)}f^*(p)\mathrm{e}^{\mathrm{i}t[pv/t-\mu(p)]}\right), \quad (15)$$

where $f^*(p)$ is assumed to be extended to $(-\infty,0)$ as an even function $f^*(p) = f^*(-p)$, $\alpha(x) = \operatorname{arccosh}(1/x) = \log\{1/x+\sqrt{x^{-2}-1}\}$.

Introducing the new integration variable $\tau = -\frac{2v}{\pi t}$, from (15) we arrive at

$$\phi(x,t) = \frac{t}{2\sqrt{2x}}\int_{-\omega(x,t)}^{0} \frac{\mathrm{d}\tau\ \psi(\tau,t)}{\sqrt{\cosh\alpha(x)-\cosh(\pi t\tau/2)}}, \quad (16)$$

where

$$\psi(\tau,t) = \int_{-\infty}^{\infty} \mathrm{d}pF_*(p)\mathrm{e}^{\mathrm{i}t\left(-\frac{\pi\tau}{2}p-\frac{1}{\cosh(\pi p)}\right)},$$

$\omega(x,t) = \frac{2\alpha(x)}{\pi t}$, $F_*(p) = \sqrt{p\tanh(\pi p)}f^*(p)$, and f^* is considered to be smooth and rapidly decreasing at infinity.

We calculate the asymptotics of $\psi(\tau,t)$ as $t\to\infty$, which is uniform over $\tau\in[-\omega(x,t),0]$. To this end, we find the stationary points of the phase function

$$\Phi(p,\tau) = -\frac{\pi\tau}{2}p - \frac{1}{\cosh(\pi p)}.$$

We must solve the equation

$$\Phi'_p(p,\tau) = -\frac{\pi\,\tau}{2} + \frac{\pi\sinh(\pi p)}{1+\sinh^2(\pi p)} = 0,$$

where τ is the parameter. For negative $\tau \in (-1,0)$, the last equation has two negative solutions $p_j(\tau)$, $j=1,2$, since the odd function $-\frac{\pi\sinh(\pi p)}{1+\sinh^2(\pi p)}$ reaches the maximum $\pi/2$ at $\pi p = -\log(\sqrt{2}+1)$. These roots coincide when $\tau=-1$, then become complex for $\tau<-1$. Thus, we have a situation of two merging stationary points for $\tau=-1$. In this case, the second derivative Φ is equal to zero at $\tau=-1$, which follows from the expression

$$\Phi''_{p^2}(p,\tau) = \pi^2\frac{\cosh(\pi p)[1-\sinh^2(\pi p)]}{[1+\sinh^2(\pi p)]^2},$$

whereas the third derivative is nonzero at this point:

$$\Phi'''_{p^3}(p,\tau) = \pi^2\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\cosh(\pi p)}{[1+\sinh^2(\pi p)]^2}\right)[1-\sinh^2(\pi p)] + \pi^2\left(\frac{\cosh(\pi p)}{[1+\sinh^2(\pi p)]^2}\right)\frac{\mathrm{d}[1-\sinh^2(\pi p)]}{\mathrm{d}p}.$$

The stationary points can be found explicitly,

$$p_j(\tau) = \frac{1}{\pi}\mathrm{arcsinh}\{\sigma_j(\tau)/\tau\} = \frac{1}{\pi}\log\left(\frac{\sigma_j(\tau)}{\tau} + \sqrt{\frac{\sigma_j^2(\tau)}{\tau^2}+1}\right), \quad j=1,2, \tag{17}$$

where $\sigma_j^2(\tau) = 1 + \left(\sqrt{1-\tau^2}\right)_j$. In formulas (17), we must distinguish the branches of the square root and arcsinh. The branches $\left(\sqrt{1-\tau^2}\right)_j$ differ by the index j and are chosen as follows. We perform cuts from the points ± 1 to $\pm\infty$, respectively, and assume that $\left(\sqrt{1-\tau^2}\right)_1|_{\tau=0} = 1$ for $j=1$, whereas $\left(\sqrt{1-\tau^2}\right)_2|_{\tau=0} = -1$ for $j=2$. Let us now define the branch $\mathrm{arcsinh}(\zeta)$. We perform cuts from $\pm\mathrm{i}$ to $\pm\mathrm{i}\infty$, respectively, assuming that $\mathrm{arcsinh}(0)=0$. It is useful to follow the change of $\sigma_j(\tau)/\tau$ when τ passes from $-\infty$ to -1 and then to -0 along the real axis. In this case, $\sigma_1(\tau)/\tau$ moves from $-\infty$ along the real axis to -1 and then, becoming complex, along the arc of the unit circle in the lower half-plane to the point $-\mathrm{i}$. Similarly, $\sigma_2(\tau)/\tau$ moves from -0 along the real axis to -1 and then, becoming complex, along the arc of unit radius in the upper half-plane to the point i. This allows us to calculate the position of $p_j(\tau)$ in the complex plane when τ goes from $-\infty$ to -1 and then to -0. We recall that if $\tau=-1$,

$$p_1(-1) = p_2(-1) = -\frac{1}{\pi}\log(\sqrt{2}+1).$$

Note that we also have ($\tau<0$)

$$p_j(\tau) = \frac{1}{\pi}\log\left(\frac{\sigma_j(\tau)}{\tau} + \sqrt{\frac{2\sigma_j(\tau)}{\tau^2}}\right), \quad j=1,2.$$

Let us note that $p_2(\tau)\to -0$ as $\tau\to -0$ and $p_1(\tau)\to-\infty$ as $\tau\to-0$.

We use the uniform version of the stationary phase method [5] for merging stationary points. Let us introduce the variable $\zeta=\zeta(p)$ according to

$$\Phi(p,\tau) = a_0(\tau) - a_1(\tau)\zeta + \frac{\zeta^3}{3}.$$

The stationary points $p_1(\tau)$ and $p_2(\tau)$ correspond to zeros

$$\Phi'_p(p,\tau) = (-a_1(\tau) + \zeta^2)\frac{\mathrm{d}\zeta}{\mathrm{d}p},$$

so that $\zeta_1 = \sqrt{a_1(\tau)}$, $\zeta_2 = -\sqrt{a_1(\tau)}$. Introduce

$$a_0(\tau) = \frac{\Phi(p_1(\tau),\tau) + \Phi(p_2(\tau),\tau)}{2}, \quad a_1(\tau) = \frac{[\Phi(p_2(\tau),\tau) - \Phi(p_1(\tau),\tau)]^{2/3}}{(4/3)^{2/3}}.$$

We find (see [5]) that $(t \to \infty)$

$$\psi(\tau,t) = \frac{\psi_a(\tau,t)}{t^{1/3}}\left(1 + O(t^{-1/3})\right),$$

$$\begin{aligned}\psi_a(\tau,t) = {} & \sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\frac{t}{2}[\Phi(p_1(\tau),\tau)+\Phi(p_2(\tau),\tau)]} \\ & \times\Bigg\{\left(F_*(p)\sqrt{\frac{-2\sqrt{a_1(\tau)}}{\Phi''_{p^2}(p,\tau)}}\Bigg|_{p=p_2(\tau)} + F_*(p)\sqrt{\frac{2\sqrt{a_1(\tau)}}{\Phi''_{p^2}(p,\tau)}}\Bigg|_{p=p_1(\tau)}\right)v(-t^{2/3}a_1(\tau)) \\ & + \frac{\mathrm{i}}{t^{1/3}}\left(F_*(p)\sqrt{\frac{-2}{\sqrt{a_1(\tau)}\Phi''_{p^2}(p,\tau)}}\Bigg|_{p=p_2(\tau)} - F_*(p)\sqrt{\frac{2}{\sqrt{a_1(\tau)}\Phi''_{p^2}(p,\tau)}}\Bigg|_{p=p_1(\tau)}\right) \\ & \times v'(-t^{2/3}a_1(\tau))\Bigg\},\end{aligned} \tag{18}$$

where $v(\,\cdot\,)$ (and $v'(\,\cdot\,)$) is the Airy function (and its derivative) having the asymptotics

$$v(z) = \frac{1}{2}\frac{\mathrm{e}^{-\frac{2}{3}z^{3/2}}}{z^{1/4}}\left(1 + O\left(z^{-3/2}\right)\right), \quad z \to \infty,$$
$$v(z) = \frac{\cos[\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}]}{(-z)^{1/4}}\left(1 + O\left((-z)^{-3/2}\right)\right), \quad z \to -\infty.$$

The asymptotics of the representation for $\phi(x,t)$ takes the form

$$\phi(x,t) = \frac{t^{2/3}}{2\sqrt{2}\sqrt{x}}\int_{-\omega(x,t)}^{0}\frac{\mathrm{d}\tau\sqrt{\cosh\alpha(x)}\psi_a(\tau,t)}{\sqrt{\cosh\alpha(x) - \cosh(\pi t\,\tau/2)}}(1 + O(t^{-1/3})). \tag{19}$$

For arbitrary $x \in [0,1]$, the expressions (18) and (19) are not simplified. However, if the variable x is not too small, namely,[3]

$$1 \geq x \geq 2\exp(-\pi t/2 - Ct^{-1+\delta})$$

for some $C > 0$ and small $\delta > 0$, the arguments of the Airy function and its derivatives are large as $t \to \infty$ and their asymptotics can be used. The expressions for $\psi_a(\tau,t)$ are simplified. The stationary points are not close and the traditional stationary phase method is used to calculate the asymptotics of $\psi(\tau,t)$. We find

$$\begin{aligned}\phi(x,t) = {} & \frac{\sqrt{\pi t}}{2\sqrt{x}}\int_{-\omega(x,t)}^{0}\frac{\mathrm{d}\tau\sqrt{\cosh\alpha(x)}(1 + O(t^{-1/3}))}{\sqrt{\cosh\alpha(x) - \cosh(\pi t\,\tau/2)}} \\ & \times\left(\frac{F_*(p_2(\tau))}{\sqrt{|\Phi''_{p^2}(p_2(\tau),\tau)|}}\mathrm{e}^{\mathrm{i}t\Phi(p_2(\tau),\tau)+\mathrm{i}\pi/4} + \frac{F_*(p_1(\tau))}{\sqrt{|\Phi''_{p^2}(p_1(\tau),\tau)|}}\mathrm{e}^{\mathrm{i}t\Phi(p_1(\tau),\tau)-\mathrm{i}\pi/4}\right).\end{aligned} \tag{20}$$

[3] In this case, we have $0 \leq -\omega(x,t) \leq -1 + Ct^{-\delta}$.

The asymptotics (20) enables us to write down a very rough estimate

$$|\phi(x,t)| \le C\frac{\sqrt{\pi t}}{2\sqrt{x}} \int\limits_{-\omega(x,t)}^{0} \frac{\mathrm{d}\tau\ \sqrt{\cosh\alpha(x)}}{\sqrt{\cosh\alpha(x) - \cosh\left(\pi t\,\tau/2\right)}} \times \left(|F_*(p_2(\tau)| + \frac{|F_*(p_1(\tau))|}{\sqrt{|\tau|}}\right). \qquad (21)$$

However, instead of (21), it is also useful to describe more precisely the behavior of ϕ as $t \to \infty$.

To simplify the integral in (20) asymptotically, we use the approach of [11], although with some variations of it.

5. Asymptotic simplification of (20) when $1 \ge x \ge 2\exp(-\pi t/2 - Ct^{-1+\delta})$

Let us represent (20) in the form of two terms

$$\phi(x,t) = \phi_1(x,t) + \phi_2(x,t),$$

where

$$\phi_1(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}} \int\limits_{-\omega(x,t)}^{0} \frac{\mathrm{d}\tau\sqrt{\cosh\alpha(x)}}{\sqrt{\cosh\alpha(x) - \cosh\left(\pi t\,\tau/2\right)}} \times \frac{F_*(p_2(\tau))}{\sqrt{|\Phi''_{p^2}(p_2(\tau),\tau)|}}\, \mathrm{e}^{\mathrm{i}t\Phi(p_2(\tau),\tau)+\mathrm{i}\pi/4}, \qquad (22)$$

and simplify the integral asymptotically as $t \to \infty$. To this end, we note that the phase function

$$\psi_1(\tau) = \Phi(p_2(\tau),\tau) = -\frac{\pi\,\tau}{2}p_2(\tau) - \frac{1}{\cosh(\pi p_2(\tau))}$$

has the first derivative

$$\psi_1'(\tau) = \frac{\mathrm{d}\Phi(p_2(\tau),\tau)}{\mathrm{d}\tau} = -\frac{\pi}{2}p_2(\tau) = 0,$$

which is zero at the end of the integration of $\tau = 0$. The second derivative

$$\psi_1''(\tau) = -\frac{\pi}{2}\frac{\mathrm{d}p_2(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=0} = -\frac{1}{4}$$

is negative at this point, since $\sigma_2(\tau) = \tau^2/2 + O(\tau^4)$ and $p_2(\tau) = \frac{\tau}{2\pi} + O(\tau^2)$. We recall that

$$\Phi''_{p^2}(p_2(\tau),\tau) = \pi^2\,\frac{\cosh(\pi p_2(\tau))[1 - \sinh^2(\pi p_2(\tau))]}{[1 + \sinh^2(\pi p_2(\tau))]^2}$$

is equal to π^2 at $\tau = 0$. However, at the end of integration in (22), $\tau = -\omega(x,t)$, we have $\cosh\alpha(x) - \cosh\left(\pi t\,\tau/2\right) = 0$, so that the traditional stationary phase method should be modified accordingly.

To this end, we introduce the new integration variable $\theta = \tau + \omega(x,t)$. We obtain

$$\phi_1(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}} \int\limits_{0}^{\omega(x,t)} \frac{\mathrm{d}\theta}{\sqrt{\theta}}\, g(\theta,\omega)\, \mathrm{e}^{\mathrm{i}t\Psi(\theta,\omega)}, \qquad (23)$$

where

$$\Psi(\theta,\omega) = \Phi(p_2(\theta-\omega),\theta-\omega),$$

$$g(\theta,\omega) = \frac{\sqrt{\cosh\alpha(x)}\sqrt{\theta}}{\sqrt{\cosh\alpha(x) - \cosh\left(\pi t[\theta-\omega(x,t)]/2\right)}}\,\frac{F_*(p_2(\tau))\,\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{|\Phi''_{p^2}(p_2(\tau),\tau)|}}\Bigg|_{\theta=\tau+\omega(x,t)}.$$

The function g is continuous on the integration interval. We will use the ideas from [11] and estimate asymptotically the integral (23) with the algebraic singularity of a slowly varying function $\frac{1}{\sqrt{\theta}}\, g(\theta,\omega)$ on the path of integration.

We introduce the new integration variable z according to

$$\Psi(\theta,\omega) - \Psi(0,\omega) = -[z^2/2 + a(\omega)z].$$

We assume that if $\theta = 0$, then $z = 0$, and if $\theta = \omega$, then $z = -a(\omega)$, where

$$a(\omega) = -\sqrt{2(\Psi(\theta,\omega) - \Psi(0,\omega))}.$$

In this case,

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}z}\right|_{z=-a} = -\left.\frac{z + a(\omega)}{\Psi'_\theta(\theta,\omega)}\right|_{z=-a} = -\frac{1}{\Psi''_{\theta^2}(\omega,\omega)\, \left.\frac{\mathrm{d}\theta}{\mathrm{d}z}\right|_{z=-a}}$$

and

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}z}\right|_{z=0} = -\frac{a(\omega)}{\Psi'_\theta(0,\omega)} > 0.$$

The integral for ϕ_1 takes the form

$$\phi_1(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}} \mathrm{e}^{\mathrm{i}t\Psi(0,\omega)} \int\limits_0^{-a(\omega)} \frac{\mathrm{d}z}{\sqrt{\theta}} \frac{\mathrm{d}\theta}{\mathrm{d}z} g(\theta,\omega)\, \mathrm{e}^{-\mathrm{i}t[z^2/2 + a(\omega)z]}. \tag{24}$$

Let us introduce the notation using the function $G(z)$,

$$z^{-1/2}G(z) = \frac{g(\theta,\omega)}{\sqrt{\theta}} \frac{\mathrm{d}\theta}{\mathrm{d}z},$$

and also set

$$G(z) = b_0(\omega) + b_1(\omega)z + z(z+a)^2 G_1(z),$$

where the last equality can be considered as the definition of G_1. We find b_0 and b_1 by setting $z = 0$ and $z = -a$,

$$b_0 = G(0) = g(0,\omega) \sqrt{\frac{\mathrm{d}\theta}{\mathrm{d}z}}\,\Bigg|_{z=0}, \qquad b_1 = \frac{G(-a(\omega)) - G(0)}{-a(\omega)}.$$

To complete the asymptotic reduction of (24), substituting $G(z)$ into the integrand, we integrate by parts over

$$\int\limits_0^{-a(\omega)} \mathrm{d}z \sqrt{z}\,(z+a)^2\, G_1(z)\, \mathrm{e}^{-\mathrm{i}t[z^2/2 + a(\omega)z]} = \frac{1}{\mathrm{i}t} \int\limits_0^{-a(\omega)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}t[z^2/2 + a(\omega)z]}\, G_2(z),$$

where $G_2(z) = G_1(z)\left(\frac{1}{2\sqrt{z}}(z+a) + \sqrt{z}\right) + \frac{\mathrm{d}G_1(z)}{\mathrm{d}z}\sqrt{z}(z+a)$. As a result, we find that

$$\begin{aligned} \phi_1(x,t) = \frac{\sqrt{\pi t}\, \mathrm{e}^{\mathrm{i}t\Psi(0,\omega)}}{2\sqrt{x}} \Bigg(b_0(\omega) \int\limits_0^{-a(\omega)} \frac{\mathrm{d}z}{\sqrt{z}}\, \mathrm{e}^{-\mathrm{i}t[z^2/2 + a(\omega)z]} \\ + b_1(\omega) \int\limits_0^{-a(\omega)} \mathrm{d}z \sqrt{z}\, \mathrm{e}^{-\mathrm{i}t[z^2/2 + a(\omega)z]} + \frac{1}{\mathrm{i}t} J_1(\omega,t) \Bigg), \end{aligned} \tag{25}$$

where

$$J_1(\omega,t) = \int\limits_0^{-a(\omega)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}t[z^2/2+a(\omega)z]}\, G_2(z).$$

The integrals in (25) are related to the Weber function (a cylindrical function) $D_r(s)$. We introduce some notation

$$W_0(s) = \int\limits_0^s \frac{\mathrm{d}z}{\sqrt{z}}\, \mathrm{e}^{-\mathrm{i}[z^2/2-sz]}$$

and

$$W_1(s) = \int\limits_0^s \mathrm{d}z\, \sqrt{z}\, \mathrm{e}^{-\mathrm{i}[z^2/2-sz]}.$$

The asymptotics of ϕ_1 takes the form ($\omega = \omega(x,t)$)

$$\phi_1(x,t) = \frac{\sqrt{\pi t}\, \mathrm{e}^{\mathrm{i}t\Psi(0,\omega)}}{2\sqrt{x}} \left(\frac{b_0(\omega)}{t^{1/4}} W_0(-\sqrt{t}a(\omega)) + \frac{b_1(\omega)}{t^{3/4}} W_1(-\sqrt{t}a(\omega)) + \frac{1}{\mathrm{i}t}\, J_1(\omega,t) \right). \quad (26)$$

5.1. Asymptotics of $\phi_2(x,t)$. Finally, we consider

$$\phi_2(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}} \int\limits_{-\omega(x,t)}^{0} \frac{\mathrm{d}\tau \sqrt{\cosh\alpha(x)}\, \mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\cosh\alpha(x) - \cosh(\pi t\,\tau/2)}} \times \frac{F_*(p_1(\tau))}{\sqrt{|\Phi''_{p^2}(p_1(\tau),\tau)|}}\, \mathrm{e}^{\mathrm{i}t\chi(\tau)}. \quad (27)$$

The phase function $\chi(\tau) := \Phi(p_1(\tau),\tau) = -\frac{\pi\tau}{2}p_1(\tau) - \frac{1}{\cosh(\pi p_1(\tau))}$ has no zeros of the first derivative on the integration interval and is monotone, however, the integrand has singularities of square root type at the ends of integration, in particular, $\sqrt{|\Phi''_{p^2}(p_1(\tau),\tau)|} \sim \sqrt{-\tau}$. Let us also note that $\chi(\tau) \sim -\frac{\tau}{2}\log(-\tau)$ as $\tau \to 0-$, whereas $\chi'(\tau) = -\frac{\pi}{2}p_1(\tau) \sim -\frac{1}{2}\log(-\tau)$ as $\tau \to 0-$.

The new integration variable $z = \chi(t)$ in (27) leads to the expression

$$\phi_2(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}} \int\limits_{-z_*}^{0} \frac{\mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}t\,z}}{\log(-\tau)\sqrt{-\tau(\tau+\omega)}}\, h(\tau,x,t),$$

where $-z_* = -z_*(x,t) := \chi(-\omega(x,t))$,

$$h(\tau,x,t) = \frac{\sqrt{\cosh\alpha(x)}\, \mathrm{e}^{-\mathrm{i}\pi/4} \log(-\tau)\sqrt{-\tau(\tau+\omega)}}{\sqrt{\cosh\alpha(x) - \cosh(\pi t\tau/2)}\chi'(\tau)} \frac{F_*(p_1(\tau))}{\sqrt{|\Phi''_{p^2}(p_1(\tau),\tau)|}}.$$

We introduce the function $H(z)$ by the equality

$$\frac{H(z)}{\log(-z)\sqrt{-z(z+z_*)}} = \frac{h(\tau,x,t)}{\log(-\tau)\sqrt{-\tau(\tau+\omega)}},$$

also implying the following representation for H:

$$H(z) = C_0(x,t) + C_1(x,t)z + (-z)(z+z_*)H_2(z).$$

It is obvious that

$$C_0(x,t) = H(0), \qquad C_1(x,t) = \frac{H(0) - H(-z_*)}{z_*}.$$

We arrive at

$$\phi_2(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}}\Bigg\{C_0(x,t)\int\limits_{-z_*}^{0}\frac{\mathrm{d}z\,\mathrm{e}^{\mathrm{i}t\,z}}{\log(-z)\sqrt{-z(z+z_*)}}$$

$$+\,C_1(x,t)\int\limits_{-z_*}^{0}\frac{\mathrm{d}z z\mathrm{e}^{\mathrm{i}tz}}{\log(-z)\sqrt{-z(z+z_*)}} + \int\limits_{-z_*}^{0}\mathrm{d}z\mathrm{e}^{\mathrm{i}tz}(-z)^{1/2}\sqrt{(z+z_*)}\frac{H_2(z)}{\log(-z)}\Bigg\}.$$

Integrating by parts in the last integral, we have

$$\frac{1}{\mathrm{i}t}\int\limits_{-z_*}^{0}\mathrm{d}\left(\mathrm{e}^{\mathrm{i}t\,z}\right)\,(-z)^{1/2}\sqrt{(z+z_*)}\frac{H_2(z)}{\log(-z)}$$

$$= -\frac{1}{\mathrm{i}t}\int\limits_{-z_*}^{0}\mathrm{d}z\,\mathrm{e}^{\mathrm{i}t\,z}\Bigg(H_2(z)\Bigg(\frac{-1}{2\log(-z)}\sqrt{\frac{z+z_*}{-z}} + \frac{1}{2\log(-z)}\sqrt{\frac{-z}{z+z_*}}$$

$$+\,\frac{1}{\log^2(-z)}\sqrt{\frac{z+z_*}{-z}}\Bigg) + \frac{\mathrm{d}H_2(z)}{\mathrm{d}z}\frac{(-z)^{1/2}\sqrt{(z+z_*)}}{\log(-z)}\Bigg).$$

The desired asymptotic estimate for ϕ_2 now takes the form ($\zeta = tz$, $\zeta_* = \zeta_*(x,t) := tz_*(x,t)$)

$$\phi_2(x,t) = \frac{\sqrt{\pi t}}{2\sqrt{x}}\Bigg\{C_0(x,t)\int\limits_{-\zeta_*(x,t)}^{0}\frac{\mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\zeta}}{[\log(-\zeta)-\log t]\sqrt{-\zeta(\zeta+\zeta_*(x,t))}}$$

$$+\,\frac{C_1(x,t)}{t}\int\limits_{-\zeta_*(x,t)}^{0}\frac{\mathrm{d}\zeta\,\zeta\,\mathrm{e}^{\mathrm{i}\zeta}}{[\log(-\zeta)-\log t]\sqrt{-\zeta(\zeta+\zeta_*(x,t))}}(1+O(1/t))\Bigg\}. \tag{28}$$

The asymptotic expressions (28) and (26) determine the required estimate for $\phi(x,t) = \phi_1(x,t) + \phi_2(x,t)$.

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