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BV-STRUCTURE ON HOCHSCHILD COHOMOLOGY FOR EXCEPTIONAL LOCAL ALGEBRAS OF QUATERNION TYPE. THE CASE OF AN EVEN PARAMETER

A. I. GENERALOV AND A. V. SEMENOV

Abstract. A full description is provided for the BV-structure on the Hochschild cohomology of exceptional local algebras of quaternion type determined by parameters $(k, 0, d)$ in the case of an even parameter $k \geq 3$, according to Erdmann's classification. The method of comparison morphisms and the weak self-homotopy method are developed and used. This article may be viewed as a generalization of similar results about the BV-structures on the Hochschild cohomology of algebras of quaternion type.

§1. Introduction

Algebras of dihedral, semidihedral, and quaternion types arise naturally from Erdmann's classical book [3] as a product of classification of tame blocks. The Hochschild cohomology of such algebras is well studied by a large body of mathematicians, such as A. I. Generalov (see [5, 6, 7]), A. A. Ivanov (see [11, 12]), C. Cibils (see [2]) and many others: see also [20] and [8], where one can find more references on studies of Hochschild cohomology. For an associative algebra A there are many structures on its Hochschild cohomology algebra $HH^*(A)$: for example, it has a graded commutative algebra structure (see [9]) and graded Lie algebra structure, introduced by Gerstenhaber in his paper [4]. T. Tradler was the first who described the BV-structure onthe Hochschild cohomology of finite dimensional symmetric algebras (see [18]). The problem here is that the BV-structure is defined in terms of the bar-resolution. It is almost impossible to compute such a structure for concrete examples, because the dimension of resolution's items grows exponentially. In order to avoid this problem, we use the method of comparison morphisms (see also [10] and [12]).

The significance of the BV-structure is in the fact that it gives a method to compute the Gerstenhaber Lie bracket, which is an important and hard-reached structure on $HH^*(A)$. In this paper we deal with the Hochschild cohomology algebra for algebras of quaternion type $R(k, 0, d)$ over an algebraically closed field K of characteristic 2, described in [8]. Some partial cases of this family of algebras were studied in [10] for the case of $R(2, 0, 0)$ and in [12] for the case of $R(k, 0, 0)$. Here we study only the case of an even parameter $k \geq 3$, because in [8] it was shown that the cases of an even and an odd parameter differ significantly.

Key words and phrases. Hochschild cohomology, homological algebra, Gerstenhaber bracket, Lie algebra, BV-structure.

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Note that the calculation of these structures is a difficult task and there are only a few examples of such calculations: for instance, one should mention the papers of Menichi (see [16, 17]), Yang (see [21]), Tradler (see [18]), Ivanov (see [12]) and two articles [19] and [14] about BV-structures of Frobenius algebras.

§2. Main definitions and facts

2.1. Hochschild (co)homology. For an associative algebra A over a field K, its nth Hochschild cohomology is the vector space $HH^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ for $n \geq 0$, where $A^e = A \otimes A^{op}$ is the enveloping algebra for A. Notice that the *bar-resolution* is a free resolution

$$A \xleftarrow{\mu=d_0} A^{\otimes 2} \xleftarrow{d_1} A^{\otimes 3} \xleftarrow{d_2} \dots \xleftarrow{d_n} A^{\otimes n+2} \xleftarrow{d_{n+1}} A^{\otimes n+3} \dots$$

with differentials

$$d_n(a_0 \otimes \dots a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}.$$

One can construct the *normalized bar-resolution*, in which the nth element is given by the formula $\overline{Bar}(A)_n = A \otimes \bar{A}^{\otimes n} \otimes A$, where $\bar{A} = A/\langle 1_A \rangle$, and the differentials are induced by those of the bar-resolution.

We define the nth Hochschild homology space $HH_n(A)$ as follows:

$$HH_n(A) = H_n(A \otimes_{A^e} Bar_\bullet(A)) \simeq H_n(A^{\otimes(\bullet+1)}),$$

where the differentials $\partial_n \colon A^{\otimes(n+1)} \longrightarrow A^{\otimes n}$ come by mapping $a_0 \otimes \dots \otimes a_n$ to

$$\sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n a_0 \otimes \dots \otimes a_{n-1}.$$

We look closely at the complex $(\mathrm{Hom}_{A^e}(Bar_\bullet(A), A),\ \delta^\bullet)$. As always,

$$HH^\bullet(A) = H^\bullet(\mathrm{Hom}_{A^e}(Bar_\bullet(A), A)) \simeq H^\bullet(\mathrm{Hom}_k(A^{\otimes \bullet}, A)),$$

and for $f \in \mathrm{Hom}_k(A^{\otimes n}, A)$ the element $\delta^n(f)$ maps $a_1 \otimes \dots \otimes a_{n+1}$ to

$$a_1 f(a_2 \otimes \dots \otimes a_{n+1}) \\ + \sum_{1}^{n} (-1)^i f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \dots \otimes a_n) a_{n+1}.$$

One can describe a cup-product on $HH^*(A)$: for classes $a \in HH^n(A)$ and $b \in HH^m(A)$ its cup-product $a \smile b \in HH^{n+m}(A)$ is defined by the class of the cup-product of the representatives $a \in \mathrm{Hom}_k(A^{\otimes n}, A)$ and $b \in \mathrm{Hom}_k(A^{\otimes m}, A)$. So, by linear extension,

$$\smile \colon HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A),$$

the Hochschild cohomology space $HH^\bullet(A) = \bigoplus_{n \geq 0} HH^n(A)$ becomes a graded commutative algebra.

2.2. Gerstenhaber bracket. For $f \in \mathrm{Hom}_k(A^{\otimes n}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes m}, A)$ one can define $f \circ_i g \in \mathrm{Hom}_k(A^{\otimes n+m-1}, A)$ by the following rules:

(1) if $n \geq 1$ and $m \geq 1$, put

$$f \circ_i g(a_1 \otimes \dots \otimes a_{n+m-1}) = f(a_1 \otimes \dots a_{i-1} \otimes g(a_i \otimes \dots a_{i+m-1}) \otimes \dots \otimes a_{n+m-1}),$$

(2) if $n \geq 1$ and $m = 0$, put

$$f \circ_i g(a_1 \otimes \dots \otimes a_{n-1}) = f(a_1 \otimes \dots a_{i-1} \otimes g \otimes a_i \otimes \dots \otimes a_{n+m-1}),$$

because $g \in A$ in this case;

(3) otherwise set $f \circ_i g = 0$.

So put $a \circ b = \sum\limits_{i=1}^{n} (-1)^{(m-1)(i-1)} a \circ_i b$.

Definition 1. For any $f \in \mathrm{Hom}_k(A^{\otimes n}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes m}, A)$ we define the *Gerstenhaber bracket of f and g* by the formula

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f.$$

This bracket obviously lies in $\mathrm{Hom}_k(A^{\otimes n+m-1}, A)$, so for $a \in HH^n(A)$ and $b \in HH^m(A)$ we can define $[a, b] \in HH^{n+m-1}(A)$ as a class of the Gerstenhaber bracket for representatives a and b. This bracket induces consistently the map

$$[\,-\,,\,-\,]\colon HH^*(A) \times HH^*(A) \longrightarrow HH^*(A),$$

which gives us the structure of a graded Lie algebra on Hochschild cohomology. Also one can show that $(HH^*(A), \smile, [-,-])$ is a Gerstenhaber algebra (see [4]).

2.3. BV-structure.

Definition 2. The *Batalin–Vilkovisky algebra* (or BV-algebra) is a Gerstenhaber algebra $(A^\bullet, \smile, [-,-])$ together with an operator $\Delta^\bullet$ of degree -1 such that $\Delta \circ \Delta = 0$ and

$$[a, b] = -(-1)^{(|a|-1)|b|}\big(\Delta(a \smile b) - \Delta(a) \smile b - (-1)^{|a|} a \smile \Delta(b)\big)$$

for homogeneous $a, b \in A^\bullet$.

Note that in this definition, we use the signs as in [18] (cf. [13], Remark 2.5).

For $a_0 \otimes \ldots \otimes a_n \in A^{\otimes(n+1)}$ define

$$\begin{aligned}\mathfrak{B}(a_0 \otimes \ldots \otimes a_n) = {} & \sum_{i=0}^{n} (-1)^{in} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1} \\ & + \sum_{i=0}^{n} (-1)^{in} a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}.\end{aligned}$$

Obviously $\mathfrak{B}(a_0 \otimes \ldots \otimes a_n) \in A^{\otimes(n+2)} \simeq A \otimes_{A^e} A^{\otimes(n+3)}$, hence it can be lifted to the chain map of complexes. Observe that $\mathfrak{B} \circ \mathfrak{B} = 0$, so one can consistently define the induced map on $HH^*(A)$.

Definition 3. The map $\mathfrak{B}\colon HH_\bullet(A) \longrightarrow HH_{\bullet+1}(A)$ defined above is called the Connes $\mathfrak{B}$-operator.

Definition 4. An algebra A is said to be symmetric if it is isomorphic (as an A^e-module) to its dual $DA = \mathrm{Hom}_K(A, K)$.

For a symmetric algebra A one can always find a nondegenerate symmetric biliniar form $\langle -, - \rangle\colon A \times A \longrightarrow K$. Obviously, the reverse statement is also true: for any such form the corresponding algebra A is symmetric. So in the case of symmetric algebras the Hochschild homology and colomology are dual to each other:

$$\begin{aligned}\mathrm{Hom}_K(A \otimes_{A^e} Bar_\bullet(A), K) &\simeq \mathrm{Hom}_{A^e}(Bar_\bullet(A), \mathrm{Hom}_K(A, K)) \\ &\simeq \mathrm{Hom}_{A^e}(Bar_\bullet(A), A),\end{aligned}$$

hence there exists an operator $\Delta\colon HH^n(A) \longrightarrow HH^{n-1}(A)$, which corresponds to the Connes $\mathfrak{B}$-operator.

So for symmetric A the algebra $HH^*(A)$ is clearly a BV-algebra (see [18] for details), and, as noticed, the Connes operator for homology corresponds to the Δ operator on cohomology.

Theorem 1 ([15, Theorem 1])**.** *The above cup-product, the Gerstenhaber bracket, and the operator* Δ *induce the structure of a BV-algebra on* $HH^*(A)$*. Furthermore, for* $f \in \mathrm{Hom}_K(A^{\otimes n}, A)$ *the element* $\Delta(f) \in \mathrm{Hom}_K(A^{\otimes(n-1)}, A)$ *can be calculated by the formula*

$$\langle \Delta(f)(a_1 \otimes \ldots \otimes a_{n-1}), a_n\rangle = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle f(a_i \otimes \ldots \otimes a_{n-1} \otimes a_n \otimes a_1 \otimes \ldots \otimes a_{i-1}), 1\rangle$$

for any $a_i \in A$*.*

Remark 1. All constructions here can be described in terms of the normalized bar-resolution.

§3. Weak self-homotopy

3.1. The resolution. Let K be an algebraically closed field of characteristic 2 and let $c, d \in K$. Define $R(k, c, d) = K\langle X, Y\rangle/I$, where I is the ideal in $K\langle X, Y\rangle$ spanned by

$$X^2 + Y(XY)^{k-1} + c(XY)^k, Y^2 + X(YX)^{k-1} + d(XY)^k, X(YX)^k, Y(XY)^k.$$

It can easily be seen that $(XY)^k + (YX)^k \in I$. Now let B be the standard basis of $R = R(k, c, d)$: we recall that

$$B = \{1, x(yx)^i, y(xy)^i, (xy)^i, (yx)^i\}_{i=0}^{k-1} \cup \{(xy)^k\}$$

(see [8]). So the set $B_1 = \{u \otimes v \mid u, v \in B\}$ is the basis for the enveloping algebra $\Lambda = R \otimes R^{op}$.

The algebras of the family $R(k, c, d)$ are symmetric, because there exists a nondegenerate symmetric bilinear form

$$\langle b_1, b_2\rangle = \begin{cases} 1, & b_1 b_2 \in Soc(R), \\ 0, & \text{otherwise.} \end{cases}$$

In order to describe the structure of a graded Lie algebra, one should only know how Δ acts on $HH^*(A)$. In this article we are interested in the case where $c = 0$, so consider only $R(k, 0, d)$: we often write R for this algebra.

Observe that the right multiplication by $\lambda \in \Lambda$ induces an endomorphism λ^* of the left Λ-module Λ. Also we will use the endomorphism of the right Λ-module Λ induced by the left multiplication by λ, which we denote by ${}^*\lambda$.

For computations we construct a 4-periodic complex in the category of (left) Λ-modules

$$P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} P_2 \xleftarrow{d_2} P_3 \xleftarrow{d_3} P_4 \xleftarrow{d_4} \ldots$$

where $P_0 = P_3 = \Lambda$, $P_1 = P_2 = \Lambda^2$ and the differentials given by the formulas

$$d_0 = \begin{pmatrix} x \otimes 1 + 1 \otimes x & y \otimes 1 + 1 \otimes y \end{pmatrix}, \quad d_1 = \begin{pmatrix} d_{11} & d_{12} \\ d_{13} & d_{14} \end{pmatrix},$$

$$d_2 = \begin{pmatrix} x \otimes 1 + 1 \otimes x \\ y \otimes 1 + 1 \otimes y + dy \otimes y + 1 \otimes dx(yx)^{k-1} + d^2 y \otimes x(yx)^{k-1} \end{pmatrix}, \quad d_3 = \lambda^*,$$

where

$$\begin{cases} d_{11} = x\otimes 1 + 1\otimes x + \sum\limits_0^{k-2} y(xy)^i \otimes y(xy)^{k-2-i}, \\ d_{12} = \sum\limits_0^{k-1} (xy)^i \otimes (yx)^{k-1-i} + d\sum\limits_0^{k-1} (xy)^i \otimes y(xy)^{k-1-i}, \\ d_{13} = \sum\limits_0^{k-1} (yx)^i \otimes (xy)^{k-1-i}, \\ d_{14} = y\otimes 1 + 1\otimes y + \sum\limits_0^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} + d\sum\limits_0^{k-1} x(yx)^i \otimes (xy)^{k-i-1}, \end{cases}$$

and

$$\lambda = \sum_0^k (xy)^i \otimes (xy)^{k-i} + \sum_1^{k-1} (yx)^i \otimes (yx)^{k-i} + \sum_0^{k-1} y(xy)^i \otimes x(yx)^{k-i-1}$$
$$+ \sum_0^{k-1} x(yx)^i \otimes y(xy)^{k-i-1} + dx(yx)^{k-1} \otimes x(yx)^{k-1}.$$

Also consider the map $\mu\colon \Lambda \longrightarrow R$ induced by multiplication: $\mu(a\otimes b) = ab$.

Theorem 2 (Proposition 3.1 in [8])**.** *The complex $P_\bullet$ equipped with the map μ forms the minimal Λ-projective resolution of R.*

It is useful to rewrite the resolution as in [10]: using the quiver Q_1 of R one can define modules $KQ_1 = \langle x, y\rangle$, and $KQ_1^* = \langle r_x, r_y\rangle$, where $r_x = x^2 + y(xy)^{k-1}$, $r_y = y^2 + x(yx)^{k-1} + d(xy)^k$. It is easy to see that

$$R\otimes KQ_1 \otimes R = R\otimes \langle x\rangle \otimes R \oplus R\otimes\langle y\rangle\otimes R \simeq R\otimes R^{op} \oplus R\otimes R^{op} = \Lambda\oplus\Lambda,$$

so we can represent the resolution $P_\bullet$ in the following form:

$$R \xleftarrow{\ \mu\ } R\otimes R \xleftarrow{\ d_0\ } R\otimes KQ_1\otimes R \xleftarrow{\ d_1\ } R\otimes KQ_1^* \otimes R \xleftarrow{\ d_2\ } R\otimes R \xleftarrow{\ d_3\ }$$

where $P_{n+4} = P_n$ for $n\in\mathbb{N}$. The differentials are given by the formulas

- $d_0(1\otimes x\otimes 1) = x\otimes 1 + 1\otimes x$, $d_0(1\otimes y\otimes 1) = y\otimes 1 + 1\otimes y$;
- $d_1(1\otimes r_x\otimes 1) = 1\otimes x\otimes x + x\otimes x\otimes 1 + \sum\limits_{i=0}^{k-2} y(xy)^i\otimes x\otimes y(xy)^{k-2-i} + \sum\limits_{i=0}^{k-1} (yx)^i \otimes y\otimes (xy)^{k-1-i}$,
  $d_1(1\otimes r_y\otimes 1) = 1\otimes y\otimes y + y\otimes y\otimes 1 + \sum\limits_{i=0}^{k-2} x(yx)^i\otimes y\otimes x(yx)^{k-2-i} + d\sum\limits_{i=0}^{k-1} x(yx)^i\otimes y\otimes (xy)^{k-1-i} + \sum\limits_{i=0}^{k-1}(xy)^i\otimes x\otimes (yx)^{k-1-i} + d\sum\limits_{i=0}^{k-1} (xy)^i\otimes x\otimes y(xy)^{k-1-i}$;
- $d_2(1\otimes 1) = x\otimes r_x\otimes 1 + 1\otimes r_x\otimes x + y\otimes r_y\otimes 1 + 1\otimes r_y\otimes y + dy\otimes r_y\otimes y + d\otimes r_y\otimes x(yx)^{k-1} + d^2 y\otimes r_y\otimes x(yx)^{k-1}$;
- $d_3 = \rho\mu$, where $\rho(1) = \sum\limits_{b\in B} b^*\otimes b + dx(yx)^{k-1}\otimes x(yx)^{k-1}$ and $\mu\colon R\otimes R\longrightarrow R$ is a multiplication map.

3.2. Construction.

Definition 5. For the complex

$$0 \longleftarrow N \xleftarrow{\ d_0\ } Q_0 \xleftarrow{\ d_1\ } Q_1 \xleftarrow{\ d_2\ } Q_2 \ldots$$

we define *the weak self-homotopy* as a collection of K-homomorphisms $t_{n+1}\colon Q_n \longrightarrow Q_{n+1}$ and $t_0\colon N\longrightarrow Q_0$ such that $t_nd_n + d_{n+1}t_{n+1} = id_{Q_n}$ for all $n\geq 0$ and $d_0t_0 = id_N$.

Now we need to construct a weak self-homotopy $\{t_i\colon P_i \longrightarrow P_{i+1}\}_{i\geq -1}$ for this projective resolution, as in [1] (here $P_{-1} = R$). In order to do this, we define a bimodule derivation $C\colon KQ \longrightarrow KQ\otimes KQ_1\otimes KQ$ by sending the path $\alpha_1\dots\alpha_n$ to $\sum\limits_{i=1}^{n}\alpha_1\dots\alpha_{i-1}\otimes\alpha_i\otimes\alpha_{i+1}\dots\alpha_n$, and consider the induced map $C\colon R\longrightarrow R\otimes KQ_1\otimes R$. So one can define $t_{-1}(1) = 1\otimes 1$ and $t_0(b\otimes 1) = C(b)$ for $b\in B$. Now construct $t_1\colon P_1\longrightarrow P_2$ by the following rules: for $b\in B$ let

$$t_1(b\otimes x\otimes 1)$$
$$=\begin{cases} 0, & bx\in B\setminus\{y(xy)^{k-1}\},\\ 1\otimes r_x\otimes 1, & b=x,\\ 1\otimes r_x\otimes x^2+x\otimes r_x\otimes x+x^2\otimes r_x\otimes 1+ & \\ (yx)^{k-1}\otimes r_y\otimes (xy)^{k-1}, & b=(xy)^k,\\ T\big(y\otimes r_x\otimes 1+(xy)^{k-1}\otimes r_x\otimes y(xy)^{k-2}+1\otimes r_y\otimes (xy)^{k-1} & \\ +dx\otimes r_x\otimes (xy)^{k-1}+dy(xy)^{k-1}\otimes r_x\otimes y(xy)^{k-2}\big), & b=Tyx,\\ y\otimes r_y\otimes 1+1\otimes r_y\otimes y+dy\otimes r_y\otimes y & \\ +d\otimes r_y\otimes x(yx)^{k-1}+d^2y\otimes r_y\otimes x(yx)^{k-1}, & b=y(xy)^{k-1},\end{cases}$$

and let

$$t_1(b\otimes y\otimes 1)$$
$$=\begin{cases} 0, & by\in B,\\ 1\otimes r_y\otimes 1, & b=y,\\ T\big(x\otimes r_y\otimes 1+(yx)^{k-1}\otimes r_y\otimes x(yx)^{k-2}+1\otimes r_x\otimes (yx)^{k-1}, & \\ +d\otimes r_x\otimes y(xy)^{k-1}+d(yx)^{k-1}\otimes r_y\otimes (xy)^{k-1}\big), & b=Txy.\end{cases}$$

In order to define $t_2\colon P_2\longrightarrow P_3$, one can put

- $t_2(x\otimes r_x\otimes 1) = 1\otimes 1$,
- $t_2(xy\otimes r_x\otimes 1) = dy(xy)^{k-1}\otimes y(xy)^{k-2}$,
- $t_2(y(xy)^{k-1}\otimes r_x\otimes 1) = 1\otimes x$,
- $t_2((xy)^k\otimes r_x\otimes 1) = \sum\limits_{i=0}^{k-1}((yx)^i\otimes y(xy)^{k-i-1}+y(xy)^i\otimes (xy)^{k-i-1})+dx(yx)^{k-1}\otimes (xy)^{k-1}$,
- $t_2((yx)^i\otimes r_x\otimes 1) = \sum\limits_{j=1}^{i-1}(yx)^j\otimes y(xy)^{i-j-1}+\sum\limits_{j=1}^{i}y(xy)^{j-1}\otimes (xy)^{i-j}+dx(yx)^{k-1}\otimes (xy)^{i-1}$ for $i>1$,
- $t_2(yx\otimes r_x\otimes 1) = y\otimes 1+dx(yx)^{k-1}\otimes 1+d(xy)^{k-1}\otimes x+\sum\limits_{i=0}^{k-2}dx(yx)^i\otimes (yx)^{k-i-1}+\sum\limits_{i=1}^{k-2}d(xy)^i\otimes x(yx)^{k-i-1}$,
- $t_2(x(yx)^i\otimes r_x\otimes 1) = \sum\limits_{j=1}^{i}((xy)^j\otimes (xy)^{i-j}+x(yx)^{j-1}\otimes y(xy)^{i-j})+\delta_{i,1}dy(xy)^{k-1}\otimes (yx)^{k-1}$,
- $t_2(b\otimes r_x\otimes 1) = 0$ otherwise;

next, let

- $t_2((xy)^i\otimes r_y\otimes 1) = \sum\limits_{j=1}^{i-1}(xy)^j\otimes x(yx)^{i-j-1}+\sum\limits_{j=1}^{i}x(yx)^{j-1}\otimes (yx)^{i-j}+d\sum\limits_{j=1}^{i}x(yx)^{j-1}\otimes y(xy)^{i-j}+d\sum\limits_{j=1}^{i-1}(xy)^j\otimes (xy)^{i-j}$ for $i>1$,
- $t_2(xy\otimes r_y\otimes 1) = x\otimes 1+dx\otimes y$,

- $t_2(y(xy)^i \otimes r_y \otimes 1) = \sum\limits_{j=1}^{i}((yx)^j \otimes (yx)^{i-j} + y(xy)^{j-1} \otimes x(yx)^{i-j}) + d\sum\limits_{j=1}^{i}((yx)^j \otimes y(xy)^{i-j} + y(xy)^{j-1} \otimes (xy)^{i-j+1}) + d^2x(yx)^{k-1} \otimes (xy)^i + dx(yx)^{k-1} \otimes x(yx)^{i-1} + \delta_{i,1} \cdot d(xy)^{k-1} \otimes y(xy)^{k-1}$ for $1 \leq i \leq k-1$,
- $t_2(b \otimes r_y \otimes 1) = 0$ otherwise.

Finally, let $t_3\colon R \otimes R \longrightarrow R \otimes R$ be the map defined by the rules $t_3((xy)^k \otimes 1) = 1 \otimes 1$ and $t_3(b \otimes 1) = 0$ elsewhere. It remains to put $t_{n+4} = t_n$ for any $n \geq 4$.

Theorem 3. *The above family of maps $\{t_i\colon P_i \longrightarrow P_{i+1}\}_{i=0}^{+\infty}$ together with $t_{-1}\colon R \longrightarrow P_0$ forms a weak self-homotopy for the resolution $P_\bullet$.*

Proof. For any $n \in \mathbb{N}$ it remains to verify the commutativity of the required diagrams, which is straight-up obvious from the definitions of t_n for $n \leq 4$ and from periodicity for $n \geq 5$. □

§4. Comparison morphisms

Consider the normalized bar-resolution $\overline{Bar}_\bullet(R) = R \otimes \bar{R}^{\otimes \bullet} \otimes R$, where $\bar{R} = R/(k \cdot 1_R)$. We now need to construct a comparison morphisms between $P_\bullet$ and $\overline{Bar}_\bullet(R)$:

$$\Phi\colon P_\bullet \longrightarrow \overline{Bar}_\bullet(R) \text{ and } \Psi\colon \overline{Bar}_\bullet(R) \longrightarrow P_\bullet.$$

It is easy to check that there exists a weak self-homotopy defined by the formula $s_n(a_0 \otimes \ldots \otimes a_n \otimes 1) = 1 \otimes a_0 \otimes \ldots \otimes a_n \otimes 1$ over $\overline{Bar}_\bullet(R)$, so one can put $\Phi_n = s_{n-1}\Phi_{n-1}d_{n-1}^P$ and $\Phi_0 = id_{R \otimes R}$.

Lemma 1. *Let $\Psi\colon \overline{Bar}_\bullet(R) \longrightarrow P_\bullet$ be the chain map constructed by using $t_\bullet$. Then for any $n \in \mathbb{N}$ and any $a_i \in R$ the following formula holds*:

$$\Psi_n(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1) = t_{n-1}(a_1\Psi_{n-1}(1 \otimes a_2 \otimes \ldots \otimes a_n \otimes 1)).$$

Proof. This follows from Lemma 2.5 of [10]. □

Let us, for example, compute first items of $\Phi_\bullet$ directly.

(1) The map Φ_1 is induced by the embedding $R \otimes kQ_1 \otimes R \longrightarrow R \otimes \bar{R} \otimes R$,

(2) $\Phi_2(1 \otimes r_x \otimes 1) = 1 \otimes x \otimes x \otimes 1 + \sum\limits_{i=0}^{k-2} 1 \otimes y(xy)^i \otimes x \otimes y(xy)^{k-i-2} + \sum\limits_{i=1}^{k-1} 1 \otimes (yx)^i \otimes y \otimes (xy)^{k-i-1}$,

$\Phi_2(1 \otimes r_y \otimes 1) = 1 \otimes y \otimes y \otimes 1 + \sum\limits_{i=0}^{k-2} 1 \otimes x(yx)^i \otimes y \otimes x(yx)^{k-i-2} + d\sum\limits_{i=0}^{k-1} 1 \otimes x(yx)^i \otimes y \otimes (xy)^{k-i-1} + \sum\limits_{i=1}^{k-1} 1 \otimes (xy)^i \otimes x \otimes (yx)^{k-i-1} + d\sum\limits_{i=1}^{k-1} 1 \otimes (xy)^i \otimes x \otimes y(xy)^{k-i-1}$,

(3) $\Phi_3(1 \otimes 1) = 1 \otimes x \otimes x \otimes x \otimes 1 + \sum\limits_{i=0}^{k-2} 1 \otimes x \otimes y(xy)^i \otimes x \otimes y(xy)^{k-i-2} + \sum\limits_{i=1}^{k-1} 1 \otimes x \otimes (yx)^i \otimes y \otimes (xy)^{k-i-1} + 1 \otimes y \otimes y \otimes y \otimes 1 + d \otimes y \otimes x(yx)^{k-1} \otimes y \otimes 1 + \sum\limits_{i=0}^{k-2} 1 \otimes y \otimes x(yx)^i \otimes y \otimes x(yx)^{k-i-2} + \sum\limits_{i=1}^{k-1} 1 \otimes y \otimes (xy)^i \otimes x \otimes (yx)^{k-i-1} + d \otimes y \otimes y \otimes y \otimes y + d^2 \otimes y \otimes x(yx)^{k-1} \otimes y \otimes y + d^2 \otimes y \otimes y \otimes y \otimes x(yx)^{k-1} + d^3 \otimes y \otimes x(yx)^{k-1} \otimes y \otimes x(yx)^{k-1}$,

(4) $\Phi_4(1 \otimes 1) = \sum_{b \in B} 1 \otimes b\Phi_3(1 \otimes 1)b^* + d \otimes x(yx)^{k-1}\Phi_3(1 \otimes 1)x(yx)^{k-1}$.

Also it is easy to see that $\Psi_0 = id_{R \otimes R}$, $\Psi_1(1 \otimes b \otimes 1) = C(b)$, $\Psi_2(1 \otimes a_1 \otimes a_2 \otimes 1) = t_1\big(a_1C(a_2)\big)$, and so on: we are only interested in the recursive structure like in Lemma 1.

In order to obtain a BV-structure on the Hochschild cohomology, one needs to compute $\Delta : HH^n(R) \longrightarrow HH^{n-1}(R)$. By the Poisson rule

$$[a \smile b, c] = [a, c] \smile b + (-1)^{|a|(|c|-1)}(a \smile [b, c]),$$

so, since $\mathrm{char} K = 2$, we obtain

$$\Delta(abc) = \Delta(ab)c + \Delta(ac)b + \Delta(bc)a + \Delta(a)bc + \Delta(b)ac + \Delta(c)ab,$$

hence we only need to know Δ on the generating elements of $HH^*(R)$ and also on the cup-products of such elements. Also for $\alpha \in HH^n(R)$ there exists a cocycle $f \in \mathrm{Hom}(P_n, R)$ such that the following formula holds: $\Delta(\alpha) = \Delta(f\Psi_n)\Phi_{n-1}$. So

$$\Delta(\alpha)(a_1 \otimes \ldots \otimes a_{n-1}) = \sum_{b\in B\setminus\{1\}} \langle \sum_{i=1}^{n} (-1)^{i(n-1)} \alpha(a_i \otimes \ldots \otimes a_{n-1} \otimes b \otimes a_1 \otimes \ldots \otimes a_{i-1}), 1\rangle b^*,$$

where $\langle b, c\rangle$ is the bilinear form defined above.

§5. BV-structure

For an algebraically closed field K of characteristic 2 consider

$$\mathcal{X} = \{p_1, p_2, p_3, p_4, q_1, q_2, w_1, w_2, w_3, e\},$$

where

$$|p_1| = |p_2| = |p_3| = |p_4| = 0,\ |q_1| = |q_2| = 1,\ |w_1| = |w_2| = |w_3| = 2,\ |e| = 4$$

and the ideal $\mathcal{I}$ in $K[\mathcal{X}]$ is spanned by the elements

- of degree 0: $p_1^k, p_2^2, p_3^2, p_4^2$ and p_ip_j for $i \neq j$;
- of degree 1: $p_3q_1 + p_2q_2,\ p_1^{k-1}q_1 + dp_3q_1,\ p_1q_2 + p_2q_1, p_1^2q_2$;
- of degree 2: $q_1q_2,\ p_1^{k-1}w_3 + dp_2w_2,\ p_2w_1, p_4w_1, p_3w_2, p_4w_2, p_4w_3,\ p_2q_1^2, p_3q_2^2,$ $p_1w_1 + p_2w_2, p_1w_1 + p_3w_3, p_1w_1 + p_4q_1^2,\ p_3w_1 + p_1w_2, p_3w_1 + p_2w_3, p_3w_1 + p_4q_2^2$;
- of degree 3: $q_1w_1 + q_2w_2,\ q_1^3 + q_2^3 + \frac{d^3(k+2)}{2}p_1q_1w_1,\ p_3q_2w_1 + p_1q_2w_2,\ p_3q_2w_1 + p_2q_2w_3,\ p_1^{k-2}q_1w_3 + dq_2w_2,\ p_1^{k-2}q_2w_3,\ p_1q_2w_1,\ p_1q_2w_3,\ q_1w_2 + q_2w_3$;
- of degree 4: $w_3^2 + p_1^2e,\ q_2^2w_1,\ q_1^2w_3,\ q_2^2w_1,\ q_2^2w_2,\ w_1^2,\ w_2^2,\ w_iw_j$ for $i \neq j$.

Theorem 4 (Theorem 2.1 in [8])**.** *$HH^*(R) \simeq \mathcal{A} = K[\mathcal{X}]/\mathcal{I}$.*

For computations one needs to know a simple form of these elements. Let P be an item of minimal projective resolution R. If $P = R \otimes R$, then denote by f the homomorphism in $\mathrm{Hom}_{R^e}(P, R)$ that sends $1 \otimes 1$ to f. If $P = R \otimes KQ \otimes R$ (or $P = R \otimes KQ_1 \otimes R$), then we denote by (f, g) the homomorphism that sends $1 \otimes x \otimes 1$ (or $1 \otimes r_x \otimes 1$) to f and $1 \otimes y \otimes 1$ (or $1 \otimes r_y \otimes 1$) to g. So one can rewrite the generating elements like in [8]:

$$\begin{cases} \text{elements of degree 0:} & p_1 = xy + yx,\ p_2 = x(yx)^{k-1},\ p_3 = y(xy)^{k-1},\ p_4 = (xy)^k, \\ \text{elements of degree 1:} & q_1 = (y(xy)^{k-2}, 1 + dy),\ q_2 = (1, d(xy)^{k-1} + x(yx)^{k-2}), \\ \text{elements of degree 2:} & w_1 = (x, 0),\ w_2 = (0, y),\ w_3 = (y, x + dxy), \\ \text{elements of degree 4:} & e = 1. \end{cases}$$

Remark 2. Also note that

$$C(b)=\begin{cases}\sum\limits_{j=0}^{i-1}(xy)^j\otimes x\otimes y(xy)^{i-j-1}+\sum\limits_{j=0}^{i-1}x(yx)^j\otimes y\otimes (xy)^{i-j-1}, & b=(xy)^i,\\ \sum\limits_{j=0}^{i}(xy)^j\otimes x\otimes (yx)^{i-j}+\sum\limits_{j=0}^{i-1}x(yx)^j\otimes y\otimes x(yx)^{i-j-1}, & b=x(yx)^i,\\ \sum\limits_{j=0}^{i}(yx)^j\otimes y\otimes (xy)^{i-j}+\sum\limits_{j=0}^{i-1}y(xy)^j\otimes x\otimes y(xy)^{i-j-1}, & b=y(xy)^i,\\ \sum\limits_{j=0}^{i-1}(yx)^j\otimes y\otimes x(yx)^{i-j-1}+\sum\limits_{j=0}^{i-1}y(xy)^j\otimes x\otimes (yx)^{i-j-1}, & b=(yx)^i.\end{cases}$$

5.1. Low degree cases. Obviously Δ is equal to zero on each combination of elements of degree zero, because it is a morphism of degree -1.

Lemma 2. *For elements of degree* 1 *in* $HH^*(R)$ *the following statements hold*: $\Delta(q_1)=\Delta(q_2)=\Delta(p_3q_2)=0$, $\Delta(p_1q_1)=dp_1$, $\Delta(p_1q_2)=\Delta(p_2q_1)=dp_2$, $\Delta(p_4q_1)=p_2$, $\Delta(p_3q_1)=\Delta(p_2q_2)=p_1^{k-1}$, $\Delta(p_4q_2)=p_3$.

Proof. We have already seen that $\Delta(a)(1\otimes 1)=\sum\limits_{b\neq 1}\langle a(C(b)),1\rangle b^*$, so we only need to compute $\langle a(C(b)),1\rangle$ on elements of degree 1. It is easy to check that

$$p_1q_1=(y(xy)^{k-1},\ dyxy),\quad p_2q_1=(0,\ x(yx)^{k-1}+d(xy)^k),$$
$$p_3q_1=(0,\ y(xy)^{k-1}),\quad p_4q_1=(0,\ (xy)^k),$$

and also one can do the same for each p_iq_2 by symmetry. Now it is clear that

$$\langle a(C(b)),1\rangle=\begin{cases}dk, & a=q_1, b=(xy)^k,\\ d(k-1), & a=p_1q_1, b\in\{(xy)^{k-1},(yx)^{k-1}\},\\ d, & a\in\{p_1q_2,p_2q_1\}, b=y,\\ 1, & a\in\{p_3q_1,p_2q_2\}, b\in\{xy,yx\}\\ & \text{ or } a=p_4q_1, b=y \text{ or } a=p_4q_2, b=x,\\ 0, & \text{otherwise,}\end{cases}$$

hence the required formulas are established. □

Lemma 3. $\Delta(x)=0$ *for any monomials* $x\in HH^2(R)$.

Proof. If $a\in HH^2(R)$, then

$$\Delta(a)(1\otimes x\otimes 1)=\Delta(a\Psi_2)\Phi_1(1\otimes x\otimes 1)=\sum_{b\neq 1}\langle (a\Psi_2)(b\otimes x+x\otimes b),1\rangle b^*,$$

$$\Delta(a)(1\otimes y\otimes 1)=\Delta(a\Psi_2)\Phi_1(1\otimes y\otimes 1)=\sum_{b\neq 1}\langle (a\Psi_2)(b\otimes y+y\otimes b),1\rangle b^*.$$

Observe that $\Psi_2(1\otimes b\otimes x\otimes 1+1\otimes x\otimes b\otimes 1)=t_1(b\otimes x\otimes 1+xC(b))$. Obviously, $\Delta(q_1q_2)=0$. Furthermore, we have

$$q_2^2=(1,\ 0),\quad q_1^2=(0,\ 1+d^2y^2+\frac{d^3k}{2}\cdot (xy)^k).$$

One needs to compute $\Psi_2(1\otimes b\otimes x\otimes 1+1\otimes x\otimes b\otimes 1)$:

(1) if $b=(xy)^i$ for $1\leq i\leq k-1$, then

$$t_1\big(b\otimes x\otimes 1+xC(b)\big)=1\otimes r_x\otimes y(xy)^{i-1}+(yx)^{k-1}\otimes r_y\otimes (xy)^{i-1}$$
$$+\delta_{i,1}\big(y(xy)^{k-2}\otimes r_x\otimes (yx)^{k-1}+dy(xy)^{k-2}\otimes r_x\otimes y(xy)^{k-1}\big),$$

(2) if $b = (yx)^i$ for $1 \le i \le k-1$, then

$$t_1\big(b \otimes x \otimes 1 + xC(b)\big) = y(xy)^{i-1} \otimes r_x \otimes 1 + (yx)^{i-1} \otimes r_y \otimes (xy)^{k-1} + \delta_{i,1}\big((xy)^{k-1} \otimes r_x \otimes y(xy)^{k-2} + dx \otimes r_x \otimes (xy)^{k-1} + dy(xy)^{k-1} \otimes r_x \otimes y(xy)^{k-2}\big),$$

(3) if $b = x(yx)^i$ for $1 \le i \le k-1$, then

$$t_1\big(b \otimes x \otimes 1 + xC(b)\big) = (xy)^i \otimes r_x \otimes 1 + x(yx)^{i-1} \otimes r_y \otimes (xy)^{k-1} + 1 \otimes r_x \otimes (yx)^i + (yx)^{k-1} \otimes r_y \otimes x(yx)^{i-1} + \delta_{i,1}\big(dx^2 \otimes r_x \otimes (xy)^{k-1} + d(xy)^k \otimes r_x \otimes y(xy)^{k-2} + dy(xy)^{k-2} \otimes r_x \otimes (xy)^k\big),$$

(4) if $b = y(xy)^{k-1}$, then

$$t_1\big(b \otimes x \otimes 1 + xC(b)\big) = y \otimes r_y \otimes 1 + 1 \otimes r_y \otimes y + dy \otimes r_y \otimes y + d \otimes r_y \otimes x(yx)^{k-1} + d^2y \otimes r_y \otimes x(yx)^{k-1},$$

(5) if $b = (xy)^k$, then

$$t_1\big(b \otimes x \otimes 1 + xC(b)\big) = x \otimes r_x \otimes x + x^2 \otimes r_x \otimes 1,$$

(6) if $b = x$ or $b = y(xy)^i$ for $0 \le i \le k-2$, then

$$t_1\big(b \otimes x \otimes 1 + xC(b)\big) = 0.$$

Now consider $\Psi_2(1 \otimes b \otimes y \otimes 1 + 1 \otimes y \otimes b \otimes 1) = t_1\big(b \otimes y \otimes 1 + yC(b)\big)$:

(1) if $b = (yx)^i$ for $1 \le i \le k-1$, then

$$t_1\big(b \otimes y \otimes 1 + yC(b)\big) = 1 \otimes r_y \otimes x(yx)^{i-1} + (xy)^{k-1} \otimes r_x \otimes (yx)^{i-1} + dx \otimes r_x \otimes x(yx)^{i-1} + dx^2 \otimes r_x \otimes (yx)^{i-1} + \delta_{i,1}\big(x(yx)^{k-2} \otimes r_y \otimes (xy)^{k-1} + d \otimes r_x \otimes x^2 + d(yx)^{k-1} \otimes r_y \otimes (xy)^{k-1}\big),$$

(2) if $b = (xy)^i$ for $1 \le i \le k-1$, then

$$t_1\big(b \otimes y \otimes 1 + yC(b)\big) = x(yx)^{i-1} \otimes r_y \otimes 1 + (xy)^{i-1} \otimes r_x \otimes (yx)^{k-1} + d(xy)^{i-1} \otimes r_x \otimes y(xy)^{k-1} + \delta_{i,1}\big((yx)^{k-1} \otimes r_y \otimes x(yx)^{k-2} + d(yx)^{k-1} \otimes r_y \otimes (xy)^{k-1}\big),$$

(3) if $b = x(yx)^{k-1}$, then

$$t_1\big(b \otimes y \otimes 1 + yC(b)\big) = y \otimes r_y \otimes 1 + 1 \otimes r_y \otimes y + dy \otimes r_y \otimes y + d \otimes r_y \otimes x(yx)^{k-1} + d^2y \otimes r_y \otimes x(yx)^{k-1},$$

(4) if $b = y(xy)^i$ for $1 \le i \le k-1$, then

$$t_1\big(b \otimes y \otimes 1 + yC(b)\big) = (yx)^i \otimes r_y \otimes 1 + y(xy)^{i-1} \otimes r_x \otimes (yx)^{k-1} + dy(xy)^{i-1} \otimes r_x \otimes y(xy)^{k-1} + 1 \otimes r_y \otimes (xy)^i + (xy)^{k-1} \otimes r_x \otimes y(xy)^{i-1} + dx \otimes r_x \otimes (xy)^i + dx^2 \otimes r_x \otimes y(xy)^{i-1},$$

(5) if $b = (xy)^k$, then

$$t_1\big(b \otimes y \otimes 1 + yC(b)\big) = y \otimes r_y \otimes y + 1 \otimes r_y \otimes x(yx)^{k-1} + dy \otimes r_y \otimes x(yx)^{k-1},$$

(6) $t_1\big(b \otimes y \otimes 1 + yC(b)\big) = 0$ otherwise.

Hence the lemma is true by given computations. □

5.2. Middle degree cases.

Lemma 4. *We have* $\Delta(q_1w_1) = \Delta(q_2w_2) = p_1^{k-2}w_3$, $\Delta(q_1w_2) = \Delta(q_2w_3) = \big(1 + d^3(l+1)p_4\big)q_1^2 + dw_2$, $\Delta(q_1w_3) = q_2^2 + dw_3$, $\Delta(q_2w_1) = q_2^2$, *where* $l = \frac{k}{2}$.

Proof. In this proof we use the delta-like function

$$\mu_{a,b} = \begin{cases} 1, & a \geq b, \\ 0, & a < b. \end{cases}$$

Fix $a \in HH^3(R)$. By the identifications $1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1 = a_1 \otimes \ldots \otimes a_n$ in the R-bimodule $R \otimes \bar{R}^{\otimes n} \otimes R$, one can observe that

$$\begin{aligned}\Delta(a)(1 \otimes r_x \otimes 1) &= \Delta(a\Psi_3)\Phi_2(1 \otimes r_x \otimes 1)\\ &= \sum_{b\neq 1}\langle (a\Psi_3)(b \otimes x \otimes x + x \otimes b \otimes x + x \otimes x \otimes b), 1\rangle b^*\\ &+ \sum_{b\neq 1}\sum_{i=0}^{k-2}\langle (a\Psi_3)(b \otimes y(xy)^i \otimes x + y(xy)^i \otimes x \otimes b + x \otimes b \otimes y(xy)^i), 1\rangle b^* \cdot y(xy)^{k-2-i}\\ &+ \sum_{b\neq 1}\sum_{i=1}^{k-1}\langle (a\Psi_3)((yx)^i \otimes y \otimes b + y \otimes b \otimes (yx)^i + b \otimes (yx)^i \otimes y), 1\rangle b^* \cdot (xy)^{k-1-i}.\end{aligned}$$

It is easy to see that

$$\Psi_3(b \otimes x \otimes x + x \otimes b \otimes x + x \otimes x \otimes b) = t_2\Big(b \otimes r_x \otimes 1 + xt_1\big(b \otimes x \otimes 1 + xC(b)\big)\Big).$$

Denote this formula by $\Psi_3(b, x)$.

- If $b = x$, then $\Psi_3(b, x) = 1 \otimes 1$,
- if $b = x(yx)^i$ for $1 \leq i \leq k-1$, then

$$\begin{aligned}\Psi_3(b, x) = \sum_{j=1}^{i} \big((xy)^j \otimes (xy)^{i-j} + x(yx)^{j-1} \otimes y(xy)^{i-j}\big) + 1 \otimes (yx)^i\\ + \delta_{i,1}\big((yx)^{k-1} \otimes (xy)^{k-1} + dy(xy)^{k-1} \otimes (yx)^{k-1} + dy(xy)^{k-1} \otimes (xy)^{k-1}\big),\end{aligned}$$

- if $b = y(xy)^{k-1}$, then $\Psi_3(b, x) = 1 \otimes x + x \otimes 1$,
- if $b = (xy)^i$ for $1 \leq i \leq k-1$, then

$$\Psi_3(b, x) = 1 \otimes y(xy)^{i-1} + \delta_{i,1}dy(xy)^{k-1} \otimes y(xy)^{k-2},$$

- if $b = (yx)^i$ for $2 \leq i \leq k-1$, then

$$\Psi_3(b, x) = \sum_{j=0}^{i-1}(yx)^j \otimes y(xy)^{i-j-1} + \sum_{j=1}^{i} y(xy)^{j-1} \otimes (xy)^{i-j} + dx(yx)^{k-1} \otimes (xy)^{i-1},$$

- if $b = yx$, then

$$\begin{aligned}\Psi_3(b, x) = y \otimes 1 + dx(yx)^{k-1} \otimes 1 + d(xy)^{k-1} \otimes x + \sum_{j=0}^{k-2} dx(yx)^j \otimes (yx)^{k-j-1}\\ + \sum_{j=1}^{k-2} d(xy)^j \otimes x(yx)^{k-j-1},\end{aligned}$$

- if $b = (xy)^k$, then $\Psi_3(b, x) = 1 \otimes y(xy)^{k-1}$,
- $\Psi_3(b, x) = 0$ otherwise.

Next for $1 \le i \le k-2$ one needs to compute

$$\Psi_3(b \otimes y(xy)^i \otimes x + y(xy)^i \otimes x \otimes b + x \otimes b \otimes y(xy)^i) \\ = t_2\Big(bt_1(y(xy)^i \otimes x \otimes 1) + y(xy)^i t_1\big(xC(b)\big) + xt_1\big(bC(y(xy)^i)\big)\Big).$$

Denote this formula by $\Psi_3(2, b, x, i)$.

- If $b = x(yx)^j$ for $1 \le j \le k-1$, $i > 0$ and $i + j \ge k$, then
$$\Psi_3(2, b, x, i) = y(xy)^{k-1} \otimes y(xy)^{i+j-k} + \delta_{i+j,k}(yx)^{k-1} \otimes y^2,$$
- if $b = y$ and $i > 0$, then
$$\Psi_3(2, b, x, i) = dy(xy)^{k-1} \otimes y(xy)^{i-1} + \delta_{i,1} d(yx)^{k-1} \otimes y^2,$$
- if $b = xy$, then

$$\Psi_3(2, b, x, i) = d\sum_{l=1}^{k-1}\big((yx)^l \otimes y(xy)^{k-1-l+i} + y(xy)^{l-1} \otimes (xy)^{k-l+i}\big) \\ + d^2x(yx)^{k-1} \otimes (xy)^{k-1+i} + \delta_{i,0}\big(1 \otimes (yx)^{k-1} + dx(yx)^{k-1} \otimes x(yx)^{k-2} \\ + d \otimes y(xy)^{k-1}\big) \\ + \begin{cases} (yx)^{k-1} \otimes (xy)^i, & i > 0 \\ \sum\limits_{l=1}^{k-1}(yx)^l \otimes (yx)^{k-1-l} + \sum\limits_{l=1}^{k-1} y(xy)^{l-1} \otimes x(yx)^{k-1-l}, & i = 0 \end{cases}$$

- if $b = (yx)^j$ for $1 \le j \le k-1$, $i + j \ge k$ and $i > 0$, then $\Psi_3(2, b, x, i) = x \otimes y(xy)^{i+j-k}$,
- $\Psi_3(2, b, x, i) = 0$ otherwise.

Only one step remains now: we need to describe

$$\Psi_3(3, b, x, i) := \Psi_3\big((yx)^i \otimes y \otimes b + b \otimes (yx)^i \otimes y + y \otimes b \otimes (yx)^i\big)$$

for $1 \le i \le k-1$.

- If $b = x(yx)^j$ for $0 \le j \le k-1$, this formula can be rewritten as:
  (1) $j = k-1$:

$$\sum_{l=1}^{i}\big((yx)^l \otimes (yx)^{i-l} + y(xy)^{l-1} \otimes x(yx)^{i-l}\big) + y \otimes x(yx)^{i-1} \\ + \delta_{i,1}\big(y \otimes x^2 + d(xy)^{k-1} \otimes x^2\big),$$

  (2) $i + j \ge k$, $j \ne k-1$:
$$y \otimes x(yx)^{i+j-k} + dx(yx)^{k-1} \otimes x(yx)^{i+j-k} + \delta_{i+j,k} d(xy)^{k-1} \otimes x^2,$$
- if $b = (yx)^j$ for $1 \le j \le k$, this formula can be rewritten as:
  (1) if $j = 1$:

$$dy(xy)^{i-1} \otimes x^2 + \delta_{i,1} dx(yx)^{k-1} \otimes (xy)^{k-1} + \delta_{i,k-1}\big(d\sum_{l=1}^{k-1}(xy)^l \otimes x(yx)^{k-l-1} \\ + d\sum_{l=1}^{k} x(yx)^{l-1} \otimes (yx)^{k-l}\big),$$

(2) if $1 < j \leq k-1$:

$$\mu_{i+j,k+1} \cdot x(yx)^{k-1} \otimes x(yx)^{i+j-k-1} + \delta_{i+j,k} \cdot \big(d \sum_{l=1}^{k} x(yx)^{l-1} \otimes (yx)^{i+j-l} + d \sum_{l=1}^{k-1} (xy)^l \otimes x(yx)^{i+j-l-1}\big) + \delta_{i+j,k+1} \cdot (xy)^{k-1} \otimes x^2,$$

(3) if $j = k$:

$$\delta_{i,1}(xy)^{k-1} \otimes x^2 + x(yx)^{k-1} \otimes x(yx)^{i-1} + \sum_{l=1}^{i} \big((yx)^l \otimes y(xy)^{i-l} + y(xy)^{l-1} \otimes (xy)^{i-l+1}\big) + dx(yx)^{k-1} \otimes (xy)^i k + d \sum_{l=1}^{k} x(yx)^{l-1} \otimes (yx)^{k-l+i} + d \sum_{l=1}^{k-1} (xy)^l \otimes x(yx)^{k+i-l-1}.$$

- if $b = y(xy)^j$ for $0 \leq j \leq k-1$, this formula can be rewritten as:
  (1) $j = 0$:

$$1 \otimes x(yx)^{i-1} + dy \otimes x(yx)^{i-1} + d^2 x(yx)^{k-1} \otimes x(yx)^{i-1} + \delta_{i,1} d^2 (xy)^{k-1} \otimes x^2,$$

  (2) $j = 1$ and $i = 1$:

$$d(xy)^{k-1} \otimes (xy)^k + d^2 y(xy)^{k-1} \otimes (xy)^k,$$

- $\Psi_3(3, b, x, i) = 0$ otherwise.

Now one should deal with $1 \otimes r_y \otimes 1$:

$$\Delta(a)(1 \otimes r_y \otimes 1) = \sum_{b \neq 1} \langle \Delta(a)(\Psi_3(b \otimes y \otimes y + y \otimes b \otimes y + y \otimes y \otimes b)), 1\rangle b^*$$

$$\sum_{b \neq 1} \sum_{i=0}^{k-2} \langle \Delta(a)(\Psi_3(b \otimes x(yx)^i \otimes y + x(yx)^i \otimes y \otimes b + y \otimes b \otimes x(yx)^i)), 1\rangle b^* x(yx)^{k-2-i}$$

$$+ d \sum_{b \neq 1} \sum_{i=0}^{k-1} \langle \Delta(a)(\Psi_3(b \otimes x(yx)^i \otimes y + x(yx)^i \otimes y \otimes b + y \otimes b \otimes x(yx)^i)), 1\rangle b^* (xy)^{k-1-i}$$

$$+ \sum_{b \neq 1} \sum_{i=1}^{k-1} \langle \Delta(a)(\Psi_3((xy)^i \otimes x \otimes b + x \otimes b \otimes (xy)^i + b \otimes (xy)^i \otimes x)), 1\rangle \times b^* ((yx)^{k-i-1} + dy(xy)^{k-i-1})$$

First, observe that

$$\Psi_3(b \otimes y \otimes y + y \otimes b \otimes y + y \otimes y \otimes b) = t_2\Big(b \otimes r_y \otimes 1 + yt_1(b \otimes y \otimes 1) + yt_1\big(yC(b)\big)\Big).$$

Denote this formula by $\Psi_3(b, y)$.

- If $b = (xy)^i$ for $1 \leq i \leq k-1$, then
  (1) if $i = 1$, then $\Psi_3(b, y) = x \otimes 1 + dx \otimes y$,

(2) if $i>1$, then

$$\Psi_3(b,y)=\sum_{l=1}^{i-1}(xy)^l\otimes x(yx)^{i-l-1}+\sum_{l=1}^{i}x(yx)^{l-1}\otimes(yx)^{i-l} + d\sum_{l=1}^{i}x(yx)^{l-1}\otimes y(xy)^{i-l}+d\sum_{l=1}^{i-1}(xy)^l\otimes(xy)^{i-l},$$

- if $b=(yx)^i$ for $1\le i\le k-1$, then:
  (1) if $i=1$, then
  $$\Psi_3(b,y)=1\otimes x+dy\otimes x+d^2(xy)^{k-1}\otimes x^2+d^2x(yx)^{k-1}\otimes x,$$
  (2) if $i>1$, then
  $$\Psi_3(b,y)=1\otimes x(yx)^{i-1}+dy\otimes x(yx)^{i-1}+d^2x(yx)^{k-1}\otimes x(yx)^{i-1}.$$
- if $b=(xy)^k$, then
  $$\Psi_3(b,y)=\sum_{l=1}^{k-1}(xy)^l\otimes x(yx)^{k-l-1}+\sum_{l=1}^{k}x(yx)^{l-1}\otimes(yx)^{k-l}.$$
- if $b{=}x(yx)^{k-1}$, then
  $$\Psi_3(b,y){=}d\sum_{l=1}^{k-1}(xy)^l\otimes x(yx)^{k-l-1}+d\sum_{l=1}^{k}x(yx)^{l-1}\otimes(yx)^{k-l}.$$
- if $b=y(xy)^i$ for $0<i\le k-1$, then

$$\Psi_3(b,y)=\sum_{l=1}^{i}\big((yx)^l\otimes(yx)^{i-l}+y(xy)^{l-1}\otimes x(yx)^{i-l}\big)+d\sum_{l=1}^{i}\big((yx)^l\otimes y(xy)^{i-l} + y(xy)^{l-1}\otimes(xy)^{i-l+1}\big)+dx(yx)^{k-1}\otimes x(yx)^{i-1}+dy\otimes(xy)^i+1\otimes(xy)^i + \delta_{i,1}\big((xy)^{k-1}\otimes(yx)^{k-1}+dy(xy)^{k-1}\otimes(yx)^{k-1}+d^2y(xy)^{k-1}\otimes y(xy)^{k-1}\big).$$

- $\Psi_3(b,y)=0$ otherwise.

Observe that if $0\le i\le k-1$, then

$$\Psi_3\big(b\otimes x(yx)^i\otimes y+x(yx)^i\otimes y\otimes b+y\otimes b\otimes x(yx)^i\big)=t_2\bigg(x(yx)^i\otimes t_1\big(yC(b)\big) + yt_1\Big(b\cdot\sum_{l=0}^{i}(xy)^l\otimes x\otimes(yx)^{i-l}+b\cdot\sum_{l=0}^{i-1}x(yx)^l\otimes y\otimes x(yx)^{i-l-1}\Big)\bigg),$$

so in order to deal with the second and third terms of the sum we only need to know the values of the obtained formula. Denote this formula by $\Psi_3(2,b,y)$.

- If $b=(xy)^j$ for $1\le j\le k$, then
  (1) if $i>0$, then

$$\Psi_3(2,b,y)=\delta_{j,k}\Big(\sum_{l=1}^{i}(xy)^l\otimes(xy)^{i-l+1}+\sum_{l=1}^{i+1}x(yx)^{l-1}\otimes y(xy)^{i-l+1}\Big) + \mu_{i+j,k}\Big(y\otimes x(yx)^{i+j-k}+dx(yx)^{k-1}\otimes x(yx)^{i+j-k}+\delta_{i+j,k}\big(d(xy)^{k-1}\otimes x^2\big)\Big),$$

  (2) if $i=0$ and $j=k$, then
  $$\Psi_3(2,b,y)=x\otimes y+y\otimes x+dx(yx)^{k-1}\otimes x+d(xy)^{k-1}\otimes x^2,$$

- if $b = yx$, then

$$\Psi_3(2,b,y) = d(xy)^i \otimes y(xy)^{k-1} + \delta_{i,0}\Big(1 \otimes (xy)^{k-1} + dy \otimes (xy)^{k-1} + d^2x(yx)^{k-1} \otimes (xy)^{k-1} + d \otimes x^2 + \sum_{l=1}^{k-1}\big((xy)^l \otimes (xy)^{k-l-1} + x(yx)^{l-1} \otimes y(xy)^{k-l-1}\big)\Big) + \mu_{i,1}\big(dy(xy)^{k-1} \otimes (yx)^i + (xy)^{k-1} \otimes (yx)^i\big),$$

- if $b = (yx)^j$ for $j \geq 2$ and $i = 0$, then

$$\Psi_3(2,b,y) = dy(xy)^{k-1} \otimes (yx)^{j-1},$$

- if $b = y(xy)^j$ for $0 \leq j \leq k-1$, then

$$\Psi_3(2,b,y) = \mu_{j,1}\delta_{i,0}\big(dy(xy)^{k-1} \otimes y(xy)^{j-1} + \delta_{j,1}d(yx)^{k-1} \otimes y^2\big) + \mu_{i+j,k-1}\big(d\sum_{l=1}^{k-1}(xy)^l \otimes x(yx)^{i+j-l} + d\sum_{l=1}^{k} x(yx)^{l-1} \otimes (yx)^{i+j+1-l}\big) + \mu_{i+j,k}\mu_{i,1}\big(x(yx)^{k-1} \otimes x(yx)^{i+j-k} + d\sum_{l=1}^{k} x(yx)^{l-1} \otimes (yx)^{i+j+1-l} + d\sum_{l=1}^{k-1}(xy)^l \otimes x(yx)^{i+j-l} + \delta_{i+j,k}(xy)^{k-1} \otimes x^2\big),$$

- if $b = x(yx)^{k-1}$, then
  (1) if $i = 0$, then $\Psi_3(2,b,y) = x \otimes 1 + 1 \otimes x$,
  (2) if $i > 0$, then

$$\Psi_3(2,b,y) = 1 \otimes x(yx)^i + \sum_{l=1}^{i}(xy)^l \otimes x(yx)^{i-l} + \sum_{l=1}^{i+1} x(yx)^{l-1} \otimes (yx)^{i-l+1},$$

- $\Psi_3(2,b,y) = 0$ otherwise.

Finally, it remains to compute

$$\Psi_3\big((xy)^i \otimes x \otimes b + x \otimes b \otimes (xy)^i + b \otimes (xy)^i \otimes x\big) = t_2\Big((xy)^i t_1\big(xC(b)\big) + xt_1\Big(bC\big((xy)^i\big)\Big)\Big)$$

for $1 \leq i \leq k-1$. Denote this by $\Psi_3(3,b,y)$.

- If $b = (xy)^j$ for $1 \leq j \leq k$ and $i + j - 1 \geq k$, then

$$\Psi_3(3,b,y) = y(xy)^{k-1} \otimes y(xy)^{i+j-k-1} + \delta_{i+j,k+1}(yx)^{k-1} \otimes y^2,$$

- if $b = y(xy)^j$ for $0 \leq j \leq k-1$ and $i + j \geq k$, then $\Psi_3(3,b,y) = x \otimes y(xy)^{i+j-k}$,
- if $b = x$, then

$$\Psi_3(3,b,y) = \delta_{i,1}dy(xy)^{k-1} \otimes y(xy)^{k-2} + 1 \otimes y(xy)^{i-1},$$

- $\Psi_3(3,b,y) = 0$ otherwise.

It remains to note that $q_1w_1 = (xy)^{k-1} = q_2w_2, q_1w_2 = y, q_1w_3 = x + dyx, q_2w_1 = x, q_2w_3 = y$. Now the required formulas follow from direct computations. □

Now we need to understand how Δ works on elements of degree 4.

Lemma 5. *$\Delta(x) = 0$ for all monomials $x \in HH^4(R)$.*

Proof. First, observe that for $a \in HH^4(R)$ the following formula holds:

$$\Delta(a)(1\otimes 1) = \Delta(a\Psi_4)\Phi_3(1\otimes 1)$$
$$= \sum_{b\neq 1}\langle (a\Psi_4)(b\cdot x\cdot x\cdot x + x\cdot b\cdot x\cdot x + x\cdot x\cdot b\cdot x + x\cdot x\cdot x\cdot b), 1\rangle b^*$$
$$+\sum_{b\neq 1}\sum_{i=0}^{k-2}\langle (a\Psi_4)(x\cdot y(xy)^i\cdot x\cdot b + y(xy)^i\cdot x\cdot b\cdot x + x\cdot b\cdot x\cdot y(xy)^i + b\cdot x\cdot y(xy)^i\cdot x), 1\rangle \times b^* y(xy)^{k-i-2}$$
$$+\sum_{b\neq 1}\sum_{i=1}^{k-1}\langle (a\Psi_4)(x\cdot (yx)^i\cdot y\cdot b + (yx)^i\cdot y\cdot b\cdot x + y\cdot b\cdot x\cdot (yx)^i + b\cdot x\cdot (yx)^i\cdot y), 1\rangle b^*(xy)^{k-i-1}$$
$$+\sum_{b\neq 1}\langle (a\Psi_4)(b\cdot y\cdot y\cdot y + y\cdot b\cdot y\cdot y + y\cdot y\cdot b\cdot y + y\cdot y\cdot y\cdot b), 1\rangle b^*(1 + dy + d^2 y(xy)^{k-1})$$
$$+\sum_{b\neq 1}\langle (a\Psi_4)(b\cdot y\cdot x(yx)^{k-1}\cdot y + y\cdot x(yx)^{k-1}\cdot y\cdot b + x(yx)^{k-1}\cdot y\cdot b\cdot y + y\cdot b\cdot y\cdot x(yx)^{k-1}), 1\rangle \times b^*(d + dy + d^3 x(yx)^{k-1})$$
$$+\sum_{b\neq 1}\sum_{i=0}^{k-2}\langle (a\Psi_4)(y\cdot x(yx)^i\cdot y\cdot b + x(yx)^i\cdot y\cdot b\cdot y + y\cdot b\cdot y\cdot x(yx)^i + b\cdot y\cdot x(yx)^i\cdot y), 1\rangle \times b^* x(yx)^{k-i-2}$$
$$+\sum_{b\neq 1}\sum_{i=1}^{k-1}\langle (a\Psi_4)(y\cdot (xy)^i\cdot x\cdot b + (xy)^i\cdot x\cdot b\cdot y + x\cdot b\cdot y\cdot (xy)^i + b\cdot y\cdot (xy)^i\cdot x), 1\rangle b^*(yx)^{k-i-1}.$$

Second, note that t_3 is not equal to zero only on $(xy)^k \otimes 1$. Denote the evaluation for the ith sum by $\Psi_4(i, b)$ for any $b \in B$. Now for the first sum we have:

1.1) if $b = xy$, then $\Psi_4(1, b) = d \otimes y(xy)^{k-2}$,
1.2) if $b = xyx$, then $\Psi_4(1, b) = d \otimes (yx)^{k-1} + d \otimes (xy)^{k-1}$,
1.3) if $b = (xy)^k$, then $\Psi_4(1, b) = 1 \otimes 1$,
1.4) $\Psi_4(1, b) = 0$ otherwise.

For the third sum and $1 \le i \le k-1$ we have:

3.1) if $b = yx$, then $\Psi_4(3, b) = d \otimes y(xy)^{i-1}$,
3.2) $\Psi_4(3, b) = 0$ otherwise.

For the fourth sum we have:

4.1) if $b = x(yx)^{k-1}$, then $\Psi_4(4, b) = d \otimes 1$,
4.2) if $b = (xy)^k$, then $\Psi_4(4, b) = 1 \otimes 1$,
4.3) $\Psi_4(4, b) = 0$ otherwise.

For the fifth sum we have:

5.1) if $b = y$, then $\Psi_4(5, b) = d \otimes 1$,
5.2) $\Psi_4(5, b) = 0$ otherwise.

It is easy to see that the other sums give zero impact, so $\Psi_4(i, b) = 0$ for any $b \in B$ and any $i \in \{2, 6, 7\}$. Now one can deduce that $\Delta(e) = \Delta(p_3 e) = 0$, $\Delta(p_2 e) = \Delta(x(yx)^{k-1}) = 0$, $\Delta(p_4 e) = 1 + 1 + dy + dy = 0$, and $\Delta(p_1^i e) = \Delta((xy)^i + (yx)^i) = 0$ for any $i \ge 1$, which yields this lemma. □

5.3. Higher degree elements. Note that if we know $\Delta(a)$ and $\Delta(b)$, then we do not need to compute $\Delta(ab)$ directly: it is sufficient to know $[a,b]$. Let a be represented by a cocycle $f\colon P_n \longrightarrow A$ and let b be represented by $g\colon P_m \longrightarrow A$. One can use the following formula:

$$[a,b] = [f\circ\Psi_n, g\circ\Psi_m]\circ\Phi_{n+m-1}.$$

Remark 3. Observe that Φ_4 can be visualized directly:

$$\Phi_4(1\otimes 1) = \sum_{b\in B} 1\otimes b\otimes x\otimes x\otimes x\otimes b^* + \sum_{b\in B}\sum_{i=0}^{k-2} 1\otimes b\otimes x\otimes y(xy)^i\otimes x\otimes y(xy)^{k-i-2}b^*$$

$$+\sum_{b\in B}\sum_{i=1}^{k-1} 1\otimes b\otimes x\otimes (yx)^i\otimes y\otimes (xy)^{k-i-1}b^* + \sum_{b\in B} 1\otimes b\otimes y\otimes y\otimes y\otimes (1+dy+d^2x(yx)^{k-1})b^*$$

$$+\sum_{b\in B}\sum_{i=0}^{k-2} 1\otimes b\otimes y\otimes x(yx)^i\otimes y\otimes x(yx)^{k-i-2}b^* + \sum_{b\in B}\sum_{i=1}^{k-1} 1\otimes b\otimes y\otimes (xy)^i\otimes x\otimes (yx)^{k-i-1}b^*$$

$$+\sum_{b\in B} 1\otimes b\otimes y\otimes x(yx)^{k-1}\otimes y\otimes (d+d^2y+d^3x(yx)^{k-1})b^* + d\otimes x(yx)^{k-1}\otimes x\otimes x\otimes x\otimes x(yx)^{k-1}$$

$$+d\otimes x(yx)^{k-1}\otimes x\otimes y(xy)^{k-2}\otimes x\otimes (xy)^k + d\otimes x(yx)^{k-1}\otimes x\otimes (yx)^{k-1}\otimes y\otimes x(yx)^{k-1}$$

$$+d\otimes x(yx)^{k-1}\otimes y\otimes y\otimes y\otimes y^2 + d^2\otimes x(yx)^{k-1}\otimes y\otimes x(yx)^{k-1}\otimes y\otimes y^2$$

$$+d\otimes x(yx)^{k-1}\otimes y\otimes (xy)^{k-1}\otimes x\otimes x(yx)^{k-1}.$$

Lemma 6. *We have* $\Delta(q_1e) = \Delta(q_2e) = 0$.

Proof. Fix $a\in HH^1(R)$. By the definitions,

$$\begin{aligned}[a,e](1\otimes 1) &= [a\circ\Psi_1, e\circ\Psi_4]\circ\Phi_4(1\otimes 1)\\ &= \big((a\Psi_1)\circ(e\Psi_4)\big)\Phi_4(1\otimes 1) + \big((e\Psi_4)\circ(a\Psi_1)\big)\Phi_4(1\otimes 1).\end{aligned}$$

What is $\Psi_4\Phi_4$? From the proofs of above lemmas, it follows that

$$\Psi_3\Phi_3(1\otimes 1) = \Psi_3(x\cdot x\cdot x) = 1\otimes 1,$$

so

$$\Psi_4\Phi_4(1\otimes 1) = \sum_{b\neq 1} t_3\big(b\Psi_3\Phi_3(1\otimes 1)b^*\big) + dt_3\big(x(yx)^{k-1}\Psi_3\Phi_3(1\otimes 1)x(yx)^{k-1}\big) = 1.$$

Finally, for $a=q_1$ or $a=q_2$ we obtain

$$\big((a\circ\Psi_1)\circ(e\circ\Psi_4)\big)\circ\Phi_4(1\otimes 1) = (a\circ\Psi_1)(1) = 0.$$

Now consider $F^u = (e\circ\Psi_4)\circ(u\circ\Psi_1) = \sum_{i=1}^{4} F_i^u$, where $F_i^u = (e\circ\Psi_4)\circ_i(u\circ\Psi_1)$. In order to compute this, we need to know $u\Psi_1(b)$ for $u=q_1$ or q_2. By direct computations we arrive at

$$q_1\Psi_1(b) = \begin{cases} y(xy)^{k-2}, & b=x,\\ 1+dy, & b=y,\\ dix(yx)^i + \delta_{i,1}\cdot y(xy)^{k-1}, & b=x(yx)^i,\ 1\le i\le k-1,\\ (xy)^i+(yx)^i+d(i+1)y(xy)^i, & b=y(xy)^i,\ 1\le i\le k-1,\\ di(xy)^i + x(yx)^{i-1}, & b=(xy)^i,\ 1\le i\le k,\\ di(yx)^i + x(yx)^{i-1}, & b=(yx)^i,\ 1\le i\le k-1,\end{cases}$$

$$q_2\Psi_1(b)=\begin{cases}1, & b=x,\\ x(yx)^{k-2}+d(xy)^{k-1}, & b=y,\\ (xy)^i+(yx)^i, & b=x(yx)^i,\ 1\le i\le k-1,\\ x(yx)^{k-1}, & b=yxy\\ y(xy)^{i-1}, & b=(xy)^i,\ 1\le i\le k,\\ y(xy)^{i-1}+\delta_{i,1}dx(yx)^{k-1}, & b=(yx)^i,\ 1\le i\le k-1,\\ 0, & \text{otherwise.}\end{cases}$$

Since $e\Psi_4(1\otimes b\otimes a_1\otimes a_2\otimes a_3\otimes 1)=et_3\Big(bt_2\Big(a_1t_1\big(a_2C(a_3)\big)\Big)\Big)$, we only need to calculate F_1^u and F_2^u on the elements $b\otimes a_1\otimes a_2\otimes a_3$ such that $a_2a_3\notin B$.

1) First consider F_1^u for $u\in HH^1(R)$. It is easy to see that

$$t_2\big(xt_1(x\otimes x\otimes 1)\big)=1\otimes 1,$$

so

$$et_3\Big(q_i\Psi_1(b)t_2\big(xt_1(x\otimes x\otimes 1)\big)\Big)b^*=et_3(q_1\Psi_1(b)\otimes 1)b^*=0$$

for any $b\in B$ and any $i\in\{1,2\}$. Hence all summands in $\Phi_4(1\otimes 1)$ involving $x\otimes x\otimes x$ give us zero. It remains to check that $t_2(yt_1(y\otimes y\otimes 1))=0$, so

$$F_1^{q_1}=F_1^{q_2}=0.$$

2) For F_2^u (where $u\in HH^1(R)$) one needs to notice that

$$t_3(bt_2(u\Psi(x)t_1(x\otimes x\otimes 1)))=t_3(bt_2(u\Psi(x)\otimes r_x\otimes 1))=0,$$

for any $b\in B$ and $u\in\{q_1,q_2\}$, so the first and eighth sums give us zero. Second, $u\Psi_1t_1(y\otimes y\otimes 1)=u\Psi_1\otimes r_y\otimes 1$, so

$$et_3\Big(bt_2\big(u\Psi_1t_1(y\otimes y\otimes 1)\big)\Big)(1+dy+d^2x(yx)^{k-1})=\begin{cases}dx(yx)^{k-2}, & u=q_2,\ b=yxy,\\ 0, & \text{otherwise.}\end{cases}$$

Obviously, another sums from $\Phi_4(1\otimes 1)$ are coming up with zeros, so

$$F_2^{q_1}=0 \text{ and } F_2^{q_2}=dx(yx)^{k-2}.$$

3) In order to compute F_3^u, notice that $\big((e\circ\Psi_4)\circ_3(u\circ\Psi_1)\big)(1\otimes b\otimes a_1\otimes a_2\otimes a_3\otimes 1)=et_3\Big(bt_2\big(a_1t_1\big(u\Psi_1(a_2)C(a_3)\big)\big)\Big)$. Then

$$t_2\big(xt_1(q_1\Psi_1(x)\otimes x\otimes 1)\big)=t_2\big(xt_1(y(xy)^{k-2}\otimes x\otimes 1)\big)=0,$$
$$t_2\big(xt_1(q_2\Psi_1(x)\otimes x\otimes 1)\big)=t_2\big(q_2\Psi_1(x)t_1(x\otimes x\otimes 1)\big),$$

and all other sums from $\Phi_4(1\otimes 1)$ give zero, what can easily be verified via the definition of t_1 and t_2, therefore in this case we can use computations of the previous case. So,

$$F_3^{q_1}=F_3^{q_2}=0.$$

4) Finally, we want to describe F_4^u. It is not hard to show that

$$\big((e\circ\Psi_4)\circ_4(u\circ\Psi_1)\big)(1\otimes b\otimes a_1\otimes a_2\otimes a_3\otimes 1)=et_3\Big(bt_2\Big(a_1t_1\big(a_2C(u\Psi_1(a_3))\big)\Big)\Big).$$

Also

$$C(u\Psi_1(x))=\begin{cases}\sum\limits_{l=0}^{k-2}(yx)^l\otimes y\otimes(xy)^{k-2-l}\\ +\sum\limits_{l=0}^{k-3}y(xy)^l\otimes x\otimes y(xy)^{k-3-l}, & u=q_1,\\ 0, & u=q_2,\end{cases}$$

and

$$C(u\Psi_1(y)) = \begin{cases} d\otimes y\otimes 1, & u=q_1,\\ \Big(\sum_{l=0}^{k-2}(xy)^l\otimes x\otimes (yx)^{k-2-l}+\sum_{l=0}^{k-3}x(yx)^l\otimes y\otimes x(yx)^{k-3-l}\Big) & \\ (1+dy)++dx(yx)^{k-2}\otimes y\otimes 1, & u=q_2.\end{cases}$$

So we have

$$t_2\bigg(xt_1\Big(xC\big(q_1\Psi_1(x)\big)\Big)\bigg) = t_2\bigg(xt_1\Big(\sum_{l=0}^{k-2}x(yx)^l\otimes y\otimes (xy)^{k-2-l}+\sum_{l=0}^{k-3}(xy)^{l+1}\otimes x\otimes y(xy)^{k-3-l}\Big)\bigg)=0,$$

and

$$t_2\bigg(xt_1\Big(y(xy)^iC\big(q_1\Psi_1(x)\big)\Big)\bigg)=\delta_{i,0}t_2\big(x\otimes r_y\otimes (xy)^{k-2}+dy(xy)^{k-1}\otimes r_x\otimes (xy)^{k-2}\big).$$

Further observe that

$$t_2\big(xt_1(q_2\Psi_1(x)\otimes x\otimes 1)\big)=t_2\big(q_2\Psi_1(x)t_1(x\otimes x\otimes 1)\big),$$

and by the definition of t_1 and t_2 one can deduce that all other sums from $\Phi_4(1\otimes 1)$ give zero impact, so this case also can be reduced to previous ones. So

$$F_4^{q_1}=0 \text{ and } F_4^{q_2}=dx(yx)^{k-2}.$$

Now we only need to collect our previous results and notice that $q_i\Delta(e)=0$ for $i\in\{1,2\}$:

$$\Delta(q_1e)(1\otimes 1)=[q_1,e](1\otimes 1)=\sum_{i=1}^{4}F_i^{q_1}=0,$$
$$\Delta(q_2e)(1\otimes 1)=[q_2,e](1\otimes 1)=\sum_{i=1}^{4}F_i^{q_2}=dx(yx)^{k-2}+dx(yx)^{k-2}=0. \qquad \square$$

Remark 4. For elements of degree six we should understand what Φ_5 is:

$$\begin{aligned}\Phi_5(1\otimes a\otimes 1) &= \sum_b 1\otimes a\otimes b\otimes x\otimes x\otimes x\otimes b^*\\
&+\sum_b\sum_{i=0}^{k-2}1\otimes a\otimes b\otimes x\otimes y(xy)^i\otimes x\otimes y(xy)^{k-2-i}b^*\\
&+\sum_b\sum_{i=1}^{k-1}1\otimes a\otimes b\otimes x\otimes (yx)^i\otimes y\otimes (xy)^{k-1-i}b^*\\
&+\sum_b 1\otimes a\otimes b\otimes y\otimes y\otimes y\otimes (1+dy+d^2x(yx)^{k-1})b^*\\
&+\sum_b\sum_{i=0}^{k-2}1\otimes a\otimes b\otimes y\otimes x(yx)^i\otimes y\otimes x(yx)^{k-2-i}b^*\\
&+\sum_b\sum_{i=1}^{k-1}1\otimes a\otimes b\otimes y\otimes (xy)^i\otimes x\otimes (yx)^{k-1-i}b^*\end{aligned}$$

$$+\sum_b 1\otimes a\otimes b\otimes y\otimes x(yx)^{k-1}\otimes y\otimes (d+d^2y+d^3x(yx)^{k-1})b^*$$
$$+\,d\otimes a\otimes x(yx)^{k-1}\otimes x\otimes x\otimes x\otimes x(yx)^{k-1}$$
$$+\,d\otimes a\otimes x(yx)^{k-1}\otimes x\otimes y(xy)^{k-2}\otimes x\otimes (xy)^k$$
$$+\,d\otimes a\otimes x(yx)^{k-1}\otimes x\otimes (yx)^{k-1}\otimes y\otimes x(yx)^{k-1}$$
$$+\,d\otimes a\otimes x(yx)^{k-1}\otimes y\otimes y\otimes y\otimes y^2$$
$$+\,d^2\otimes a\otimes x(yx)^{k-1}\otimes y\otimes x(yx)^{k-1}\otimes y\otimes y^2$$
$$+\,d\otimes a\otimes x(yx)^{k-1}\otimes y\otimes (xy)^{k-1}\otimes x\otimes x(yx)^{k-1}.$$

Lemma 7. *For all $v\in\{w_1,w_2,w_3\}$ and $e\in HH^4(R)$ all the brackets $[v,e]$ are equal to zero.*

Proof. For $v\in HH^2(R)$ and $e\in HH^4(R)$ we have
$$[v,e](1\otimes a\otimes 1)=\big((v\Psi_2)\circ(e\Psi_4)\big)\Phi_5(1\otimes a\otimes 1)+\big((e\Psi_4)\circ(v\Psi_2)\big)\Phi_5(1\otimes a\otimes 1).$$
Now we need to show that the first summand here is equal to zero. It is obvious that $\big((v\Psi_2)\circ(e\Psi_4)\big)\Phi_5=\big((v\Psi_2)\circ_1(e\Psi_4)\big)\Phi_5+\big((v\Psi_2)\circ_2(e\Psi_4)\big)\Phi_5$. Let us denote these summands by S_1^v and S_2^v respectively. One can quickly show that S_2^v is equal to zero for all v. Indeed,
$$S_2^v(a_1\otimes\ldots\otimes a_5)=vt_1(a_1C(et_3(a_2t_2(a_3t_1(a_4C(a_5)))))),$$
and since $C(1)=0$ and $et_3(b\otimes 1)=\begin{cases}1, & b=(xy)^k,\\ 0, & \text{otherwise}\end{cases}$, the required formula holds true.

One can prove that S_1^v equals zero on all summands from $\Phi_5(1\otimes a\otimes 1)$ except the fourth and seventh. Finally,

$$S_1^v\big(1\otimes a\otimes (xy)^k\otimes y\otimes x(yx)^{k-1}\otimes y\otimes (d+d^2y+d^3x(yx)^{k-1})\big)$$
$$=\begin{cases}dy+d^2x(yx)^{k-1}, & a=y,\ v=w_2,\\ dx, & a=y,\ v=w_3,\\ 0, & \text{otherwise},\end{cases}$$
$$S_1^v\big(1\otimes a\otimes (xy)^k\otimes y\otimes y\otimes y\otimes (1+dy+d^2x(yx)^{k-1})\big)$$
$$=\begin{cases}dy+d^2x(yx)^{k-1}, & a=y,\ v=w_2,\\ dx, & a=y,\ v=w_3,\\ 0, & \text{otherwise},\end{cases}$$
and for other combinations of these sums S_1^v is equal to zero. So
$$S_1^v(a_1\otimes\ldots\otimes a_5)=0$$
for any $a_1,\ldots,a_5\in B$, and hence $\big((v\Psi_2)\circ(e\Psi_4)\big)\Phi_5=0$.

It remains to consider $\big((e\Psi_4)\circ(v\Psi_2)\big)\Phi_5=\sum_{i=1}^{4}\big((e\Psi_4)\circ_i(v\Psi_2)\big)\Phi_5$. Denote $\big((e\Psi_4)\circ_i(v\Psi_2)\big)\Phi_5$ by F_i^v. It is easy to see that $F_i^v(a_1\otimes\ldots\otimes a_5)$ equals zero for any $i\in\{1,2,4\}$ on any summand of Φ_5, except maybe the first, fourth, eighth, and eleventh summands, because otherwise $t_1(a_4C(a_5))=0$.

1) Obviously, $t_2\big(yt_1(y\otimes y\otimes 1)\big)=0$, so for F_1^v it remains to investigate only the first and eighth summands. The computations show that
$$F_1^v(1\otimes a\otimes b\otimes x\otimes x\otimes x\otimes b^*)$$

$$= \begin{cases} et_3\Big(vt_1\big(x\otimes x\otimes y(xy)^{i-1}+y(xy)^{k-1}\otimes y\otimes (xy)^{i-1}\big)\otimes 1\Big)(xy)^{k-i}, & b=(xy)^i \text{ and } a=x,\\ et_3\Big(vt_1\big(y\otimes y\otimes x(yx)^{i-1}+y^2\otimes x\otimes (yx)^{i-1}\big)\otimes 1\Big)(yx)^{k-i}, & b=(yx)^i \text{ and } a=y,\\ et_3\big((yx)^k\otimes 1\big)y(xy)^{k-2}, & v=w_2,\ b=xyx \text{ and } a=x,\\ et_3\big((xy)^k\otimes 1\big)x(yx)^{k-2}, & v=w_1,\ b=yxy \text{ and } a=y,\\ et_3\big((xy)^k\otimes 1\big), & v=w_2,\ b=(xy)^k \text{ and } a=y\\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & v=w_1,\ b=(xy)^k \text{ and } a=x,\\ d(xy)^{k-1}, & v=w_3,\ b=xy \text{ and } a=x,\\ d(yx)^{k-1}, & v=w_1,\ b=yx \text{ and } a=y,\\ y(xy)^{k-2}, & v=w_2,\ b=xyx \text{ and } a=x,\\ x(yx)^{k-2}, & v=w_1,\ b=yxy \text{ and } a=y,\\ 1, & v=w_2,\ b=(xy)^k \text{ and } a=y,\\ 0, & \text{otherwise.} \end{cases}$$

So

$$F_1^{w_1} = \begin{cases} 1, & a=x,\\ x(yx)^{k-2}+d(yx)^{k-1}, & a=y, \end{cases} \quad F_1^{w_2} = \begin{cases} y(xy)^{k-2}, & a=x,\\ 1, & a=y, \end{cases}$$

$$F_1^{w_3} = \begin{cases} d(xy)^{k-1}, & a=x,\\ 0, & a=y. \end{cases}$$

2) For F_2^v we need to examine only the summands with numbers one, four, eight, and eleven from Φ_5. So for the first summand

$$F_2^v(1\otimes a\otimes b\otimes x\otimes x\otimes x\otimes b^*)$$

$$= \begin{cases} et_3\big(at_2(x^3\otimes r_x\otimes 1)\big), & v=w_1 \text{ and } b=(xy)^k,\\ et_3\big(at_2(xyx\otimes r_x\otimes 1)\big), & v=w_3 \text{ and } b=(xy)^k,\\ et_3\big(at_2((xy)^k\ r_x\otimes 1)\big)y(xy)^{k-2}, & v=w_2 \text{ and } b=xyx,\\ et_3\big(ay\otimes x+dax(yx)^{k-1}\otimes x+day(xy)^{k-1}\otimes (yx)^{k-1}\big), & v=w_3 \text{ and } b=y(xy)^{k-1},\\ et_3\big(at_2(y^2\otimes r_x\otimes 1)\big)(yx)^{k-1}, & v=w_3 \text{ and } b=yx,\\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & v=w_1,\ b=(xy)^k \text{ and } a=x,\\ d(yx)^{k-1}, & v=w_3,\ b=(xy)^k \text{ and } a=x,\\ y(xy)^{k-2}, & v=w_2,\ b=xyx \text{ and } a=x,\\ d(yx)^{k-1}, & v=w_3,\ b=y(xy)^{k-1} \text{ and } a=x,\\ d(yx)^{k-1}, & v=w_3,\ b=yx \text{ and } a=x,\\ 0, & \text{otherwise,} \end{cases}$$

and for the forth summand

$$F_2^v\big(1\otimes a\otimes b\otimes y\otimes y\otimes y\otimes (1+dy+d^2x(yx)^{k-1})b^*\big)$$

$$= \begin{cases} et_3\big(at_2((xy)^k\otimes r_y\otimes 1)\big)(x(yx)^{k-2}+d(yx)^{k-1}), & v=w_1,\ a=y \text{ and } b=yxy,\\ et_3\big(at_2((xy)^k\otimes r_y\otimes 1)\big)(1+dy+d^2x(yx)^{k-1}), & v=w_2,\ a=y \text{ and } b=(xy)^k,\\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x(yx)^{k-2}, & v=w_1,\ a=y \text{ and } b=yxy,\\ 1, & v=w_2,\ a=y \text{ and } b=(xy)^k,\\ 0, & \text{otherwise.} \end{cases}$$

So the eighth and eleventh summands give us zero and hence

$$F_2^{w_1} = \begin{cases} 1, & a = x, \\ x(yx)^{k-2}, & a = y, \end{cases} \quad F_2^{w_2} = \begin{cases} y(xy)^{k-2}, & a = x, \\ 1, & a = y, \end{cases}$$

$$F_2^{w_3} = \begin{cases} d(yx)^{k-1}, & a = x, \\ 0, & a = y. \end{cases}$$

3) In the case of F_3^v, if $t_1\big(a_3C(a_4)\big) = 0$, then the summand of the form

$$\sum 1 \otimes a_1 \otimes \ldots \otimes a_5 \otimes a_6$$

give us zero. Also if $(a_3, a_4, a_5) = (x, x, x)$, then

$$t_1\big(vt_1(x \otimes x \otimes 1) \otimes x \otimes 1\big) = \begin{cases} 1 \otimes r_x \otimes 1, & v = w_1, \\ 0, & v \neq w_1 \end{cases}$$

and

$$F_3^{w_1}(1 \otimes x \otimes (xy)^k \otimes x \otimes x \otimes x \otimes 1) = et_3\big((xy)^k \otimes 1\big) = 1.$$

There is a zero in all other cases. For the forth sum $F_3^{w_1} = 0$, and if $v = w_2$, then

$$F_3^v(1 \otimes a \otimes b \otimes y \otimes y \otimes y \otimes 1)$$
$$= \begin{cases} et_3\big((xy)^k \otimes 1\big)(1 + dy), & v = w_2,\ b = (xy)^k \text{ and } a = y, \\ et_3\big(at_2((yx)^k \otimes r_y \otimes 1)\big), & v = w_3,\ b = y(xy)^{k-1}, \\ et_3\big(d(xy)^k \otimes (yx)^{k-1} + d^2(xy)^k \otimes y(xy)^{k-1}\big), & v = w_3,\ b = (xy)^k \text{ and } a = x, \end{cases}$$

so if $b = (xy)^k$ for $v = w_2$ or if $b = y(xy)^{k-1}$ for $v = w_3$, this sum equals $(1 + dy)(1 + dy + d^2x(yx)^{k-1}) = 1$ for $a = y$. If $b = (xy)^k$ for $v = w_3$, this sum equals $d(yx)^{k-1}$ for $a = x$. At the seventh summand $F_3^v = 0$ if $v \neq w_3$. And in the case of $v = w_3$ we have

$$F_3^{w_3}\big(1 \otimes a \otimes b \otimes y \otimes x(yx)^{k-1} \otimes y \otimes (d + d^2y + d^3x(yx)^{k-1})\big)$$
$$= \sum_b et_3\Big(at_2\big(bt_1(xy \otimes y \otimes 1 + yx \otimes y \otimes 1 + 2dyxy \otimes y \otimes 1)\big)\Big)(d + d^2y + d^3x(yx)^{k-1})$$
$$= F_3^{w_3}\big(1 \otimes a \otimes b \otimes y \otimes y \otimes y \otimes (1 + dy + d^2x(yx)^{k-1})\big),$$

so we do not need to compute it, because these two summands kill each other. All other combinations give us zero, so the eighth and eleventh sums also give us zero. So

$$F_1^{w_1} = \begin{cases} 1, & a = x, \\ 0, & a = y, \end{cases} \quad F_1^{w_2} = \begin{cases} 0, & a = x, \\ 1, & a = y, \end{cases}$$

$$F_1^{w_3} = \begin{cases} d(yx)^{k-1} + d(yx)^{k-1}, & a = x, \\ x + x, & a = y \end{cases} = 0.$$

4) Finally we need to know the evaluation of F_4^v at the first, forth, eighth, and eleventh summands from Φ_5. It is obvious that for the first summand $F_4^v = 0$ if $v \neq w_1$. But $F_4^{w_1}(1 \otimes a \otimes b \otimes x \otimes x \otimes x \otimes 1) = et_3(at_2(b \otimes r_x \otimes 1)) = F_3^{w_1}(1 \otimes a \otimes b \otimes x \otimes x \otimes x \otimes 1)$, so the first summand gives $\delta_{a,x}1$, and $F_4^{w_1}$ is equal to zero for the eighth sum. Further $F_4^{w_2}(1 \otimes a \otimes b \otimes y \otimes y \otimes y \otimes 1) = et_3(at_2(b \otimes r_y \otimes 1)) = F_3^{w_2}(1 \otimes a \otimes b \otimes y \otimes y \otimes y \otimes 1)$, so $F_4^{w_2}$ gives $\delta_{a,y}1$ at the forth sum and F_4^v gives zero $v \neq w_2$ for the forth and eighth sums. So

$$F_1^{w_1} = \begin{cases} 1, & a = x, \\ 0, & a = y, \end{cases} \qquad F_1^{w_2} = \begin{cases} 0, & a = x, \\ 1, & a = y, \end{cases} \quad F_1^{w_3} = 0.$$

It remains to compute $[v,e]=\sum_{i=1}^{2} S_i^v+\sum_{i=1}^{4} F_i^v=0$ for any $v\in\{w_1,w_2,w_3\}$, hence the required formulas hold true. □

Corollary 5. *We have* $\Delta(ve)=0$ *for any* $v\in\{w_1,w_2,w_3\}$.

Proof. $\Delta(v)=0$ for any $v\in\{w_1,w_2,w_3\}$ by Lemma 3, and $\Delta(e)=0$ by Lemma 5. So by Tradler's equation we have

$$\Delta(ve)=\Delta(v)e+v\Delta(e)+[v,e]=0\cdot e+v\cdot 0+0=0$$

for $v\in\{w_1,w_2,w_3\}$. □

§6. Main result

Theorem 6. *Let* $R=R(k,0,d)$ *over an algebraically closed field* K *of characteristic* 2, *and let* Δ *be the* BV*-operator from Theorem* 1. *Then*

(1) Δ *is equal to* 0 *at the generators of* $HH^*(R)$ *from the set* $\mathcal{X}$;
(2) Δ *satisfies the relations*
- *in degree* 1: $\begin{cases}\Delta(p_1q_1)=dp_1, \Delta(p_1q_2)=\Delta(p_2q_1)=dp_2, \Delta(p_4q_1)=p_2,\\ \Delta(p_3q_1)=\Delta(p_2q_2)=p_1^{k-1},\ \Delta(p_4q_2)=p_3;\end{cases}$
- *in degree* 3 : $\begin{cases}\Delta(q_1w_1)=\Delta(q_2w_2)=p_1^{k-2}w_3,\\ \Delta(q_1w_2)=\Delta(q_2w_3)=(1+d^3(l+1)p_4)q_1^2+dw_2,\\ \Delta(q_2w_1)=q_2^2,\ \Delta(q_1w_3)+\Delta(q_2w_1)=dw_3.\end{cases}$

(3) $\Delta(ab)=0$ *at all other combinations of generators* $a,b\in\mathcal{X}$.

Proof. The identities of degree 1 come from Lemma 2 and the identities of degree 3 come from Lemma 4. Statements 1 and 3 now come from Lemmas 3–6 and Corollary 5 and it is clear from the lemmas above that there are no other conditions for Δ. □

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St. Petersburg State University, Universitetskaya nab., 7-9, St. Petersburg, 199178 Russia

Email address: ageneralov@gmail.com

Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O., 29B, St. Petersburg 199178 Russia

Email address: asemenov.spb.56@gmail.com

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