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Tight wavelet frames on the space of M-positive vectors

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Wavelets on the sets of M-positive vectors in the Euclidean space are studied. These sets are multidimensional analogs of the half-line in the Walsh analysis. Following the ideas of the Walsh analysis, the space of M-positive vectors is equipped with a coordinate-wise addition. Harmonic analysis on this space is also similar to the Walsh harmonic analysis, and the Fourier transform is such that there exists a class of so-called test functions (with a compact support of the function itself and of its Fourier transform). Tight wavelet frames consisting of the test functions are studied. A complete description of the masks generating such frames is given, and an algorithmic method for constructing them is developed. These frames may be very useful for applications to signal processing because some examples of such systems on the half-line were already investigated in this aspect, and it appeared that they have an advantage over classical wavelet systems when used for processing fractal signals and images.

Keywords: M-positive vectors; Walsh function; test-function; tight wavelet frame; refinable function.

Mathematics Subject Classification 2020: 42C10, 42C15, 42C40, 68U10

1. Introduction

The set/space of M-positive vectors in $\mathbb{R}^d$, associated with a dilation matrix M, was introduced and studied in [12, 26]. Many of such sets have a complicated fractal structure. The sets of M-positive vectors are equipped with the coordinate-wise addition modulo M, which makes them very similar to the Vilenkin groups, although they are not groups. What makes this similarity possible is that on the one hand, the functions $\chi(\cdot, \omega)$ play the role of the characters for Vilenkin groups, and on the other hand, the Walsh functions, playing the role of harmonics, are defined via these pseudo-characters.

Also, similarly to the Vilenkin groups, the concept of Fourier transform is associated with these "characters". The corresponding harmonic analysis was developed in [26], where the Poisson summation formula, the Plancherel theorem, the inversion formula, the Vilenkin–Chrestenson formulas and other natural properties of the Fourier transform and Walsh functions were proved. Another similarity between the spaces of M-positive vectors and Vilenkin groups is the existence of a class of compactly supported functions whose Fourier transform also has compact support (called test functions), which never happens in classical harmonic analysis (see, e.g., [15]).

In this paper we develop a method for constructing tight wavelet frames consisting of the test functions. Problems of this kind have been actively studied for the Vilenkin groups (see, e.g., [4, 5, 10, 18, 20, 24]), where the construction of frames is based on a refinable function in the same way as in the usual wavelet theory on the real line. An algorithmic method for constructing compactly supported tight wavelet frames on the basis of a Walsh polynomial (which plays the same role as a trigonometric polynomial in the classical analysis) was known due to [8, 25]. However it was not clear what Walsh polynomials provide the step wavelet functions, although a lot of examples were known. The answer to this question for the Vilenkon groups was given in [11]. Now we obtain similar results for the spaces of M-positive vectors.

The simplest special case of the space of M-positive vectors is the half-line, where the matrix dilation is just a positive integer M. Wavelet theory on the half-line, equipped with a coordinate-wise addition modulo M, was developed in [3, 6, 7, 23]. In particular, a number of examples of tight wavelet frames consisting of the test functions were presented there. Wavelet systems consisting of the test functions are of interest not only in the theoretical aspect, but also for some applications, in particular, for signal processing. It is shown in [7] that using some such systems on the half-line for processing fractal signals and images has an advantage over classical wavelet systems.

Note that study of wavelet frames defined on the whole space $\mathbb{R}^d$ is an old topic (see, e.g., [16]), nevertheless, it is actively investigated up to now (see, e.g., [1, 19]). We are interested in wavelets defined on the spaces of M-positive vectors, that is an essentially new topic.

2. Space of M-Positive Vectors

In this section, all the necessary information about the spaces of M-positive vector is given, and the proofs of all formulated statements can be found in [12].

Let $\langle\cdot,\cdot\rangle$ be the inner product in a Hilbert space, and μ be the Lebesgue measure on $\mathbb{R}^d$. As usual, $\mathbb{Z}^d$ is the integer lattice of $\mathbb{R}^d$ and $\mathbf{0}$ is the zero vector in $\mathbb{R}^d$. The characteristic function of a set $Y \subset \mathbb{R}^d$ is denoted by $\mathbf{1}_Y$. Let $\mathfrak{M}_d$ denote the class of dilation matrices of order d, i.e. integer $(d \times d)$-matrices M such that all their eigenvalues are bigger than 1 in absolute value. It is well known that

$$\sum_{n=1}^{\infty} \|M^{-n}\|^{\delta} < \infty \tag{1}$$

for any $M \in \mathfrak{M}_d$ and any $\delta > 0$ (see, e.g., [21, §2.2]).

Let $M \in \mathfrak{M}_d$, $m := |\det M|$ and M^* be the matrix transposed to M. Vectors $l, k \in \mathbb{Z}^d$ are said to be *congruent modulo* M (write $l \equiv k \,(\mathrm{mod}\, M)$) if $l - k = Ms$ for some $s \in \mathbb{Z}^d$. A set $D = D(M) = \{s_0, s_1, \ldots, s_{m-1}\}$, where $s_i \in \mathbb{Z}^d$, $s_0 = \mathbf{0}$ and $s_i \not\equiv s_j \,(\mathrm{mod}\, M)$ whenever $i \neq j$, $i, j \in \{0, 1, \ldots, m-1\}$, is called a *set of digits* for M. Similarly, $D^* = D(M^*) = \{s_0^*, s_1^*, \ldots, s_{m-1}^*\}$ denotes a set of digits for M^*.

Let D be a set of digits for $M \in \mathfrak{M}_d$. Denote by $X^+ = X^+(M, D)$ the set of vectors $x \in \mathbb{R}^d$ represented as

$$x = \sum_{j=-\infty}^{\infty} M^{-j} x_j = \sum_{j=-N}^{\infty} M^{-j} x_j, \quad x_j \in D, \tag{2}$$

where $N = N(x) \in \mathbb{Z}$, i.e. $x_j = \mathbf{0}$ whenever $j < -N$. Denote also by U^+ the set of vectors $x \in \mathbb{R}^d$ represented as

$$x = \sum_{j=1}^{\infty} M^{-j} x_j, \quad x_j \in D. \tag{3}$$

It is known that the Lebesgue measure of U^+ is a positive integer, see [13].

An element x given by (2) can be also written as follows:

$$x = x_{-N} x_{-N+1} \ldots x_0, x_1 x_2 \ldots,$$

that will be used in what follows. If x has a finite representation (2), i.e. $x_j = \mathbf{0}$ whenever $j > n$, we will write

$$x = x_{-N} x_{-N+1} \ldots x_0, \quad x_1 \ldots x_n,$$

Denote by $X^0 = X^0(M, D)$ the set of vectors $x \in X^+$ for which a finite representation (2) exists.

Theorem 1. *Representation* (2) *is unique for almost all* $x \in X^+(M, D)$, *and for every* $x \in X^0(M, D)$ *there exists a unique finite representation* (2).

We will use the finite representation of a vector $x \in X$ if it exists.

Definition 2. Given a matrix $M \in \mathfrak{M}_d$ and a set of its digits D, the space of *M-positive vectors* is defined by

$$X = X(M, D) := (X^+ \setminus \tilde{X}) \cup X^0,$$

where $\widetilde{X}$ is the set of vectors $x \in X^+$ that have at least two infinite representations (2).

Thus, the set of M-positive vectors is more or less similar to the set of non-negative numbers. Also we introduce the following analogs of the unit interval $[0, 1)$, of the set of non-negative integers and of the set of n-digit numbers.

$$U = U(M, D) := \{x \in X(M, D) : x_j = \mathbf{0},\ \forall\, j \le 0\},$$
$$H = H(M, D) := \{x \in X(M, D) : x_j = \mathbf{0},\ \forall\, j > 0\},$$
$$H_n = H_n(M, D) := \{x \in H(M, D) : x_j = \mathbf{0},\ \forall\, j \le -n\}.$$

It is known that U is a bounded set, has a non-empty interior (see [21,§ 2.8; 16, § 6.1; 2, 17]) and, obviously,

$$MU = \bigcup_{s \in D} (U + s). \tag{4}$$

The space X^* and the sets U^*, H^*, H_n^* are defined similarly with respect to the matrix M^*. Note that the sets U, U^* have a complex fractal structure for some matrices M, see, e.g., Fig. 1.

Note that, because of (4) and a similar relation for U^*, we have

$$M^n(U) = \bigcup_{\gamma \in H_n} (U + \gamma), \quad (M^*)^n(U^*) = \bigcup_{\gamma^* \in H_n^*} (U^* + \gamma^*). \tag{5}$$

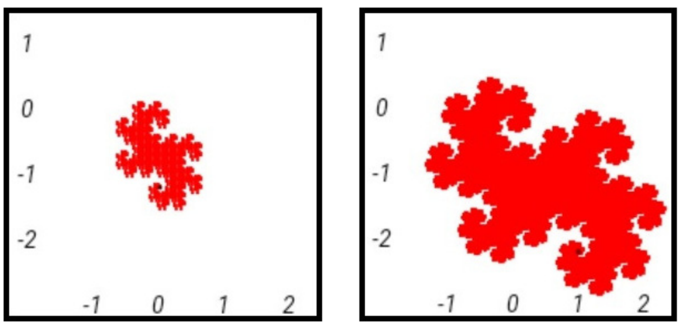
To numerate the elements of H by the non-negative integers, we set

$$\gamma_{[k]} := \sum_{j=0}^{\infty} M^j s_{k_j}, \quad \text{where } k = \sum_{j=0}^{\infty} k_j m^j, \quad k_j \in \{0, 1, \ldots, m-1\},$$

and

$$U_{n,k} := M^{-n}\gamma_{[k]} + M^{-n}(U), \quad n \in \mathbb{Z}, \quad k \in \mathbb{Z}_+. \tag{6}$$

The elements $\gamma_{[k]}^*$ and the sets $U_{n,k}^*$ are defined similarly.



(a) (b)

Fig. 1. $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$: (a) U for $D = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$; (b) U for $D = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.

Next we introduce an operation $\oplus$ of coordinate-wise addition on X. Fixing a set of digits D, we define the sum of vectors $x, y \in X$ as

$$x \oplus y := \sum_{j=-\infty}^{\infty} M^{-j}(x_j \oplus y_j),$$

where x_j, y_j are the digits from (2) for x and y, respectively, and $x_j \oplus y_j$ is the digit congruent to $x_j + y_j$ modulo M.

Proposition 3. *If $x, y \in X$, then $x \oplus y \in X$.*

For every $x \in X$ and $\omega \in X^*$, we set

$$\chi(x, \omega) := \exp\left(2\pi i \sum_{j\in\mathbb{Z}} \langle M^{-1}x_j,\, \omega_{1-j}\rangle\right) \tag{7}$$

and note the following trivial properties of these functions:

$$\chi(x \oplus y, \omega) = \chi(x, \omega)\cdot\chi(y, \omega) \quad \text{for all } x, y \in X, \quad \omega \in X^*,$$

$$\chi(Mx, \omega) = \chi(x, M^*\omega) \quad \text{for all } x \in X, \quad \omega \in X^*.$$

Note that changing roles of X and X^*, we can consider a similar function $\chi^*(\omega, x)$, and obviously, $\chi^*(\omega, x) = \chi(x, \omega)$.

Lemma 4. *If $\omega \in H^*$, $\omega \neq \mathbf{0}$, then*

$$\int_U \chi(x, \omega)\, d\mu(x) = 0. \tag{8}$$

Thus, we see that these functions χ have the same properties as the characters of a group, although our X is not a group, and a concept of the Fourier transform can be introduced in a similar way to groups.

Definition 5. For a function $f \in L^1(X)$, its *Fourier transform* is defined by

$$\widehat{f}(\omega) = \frac{1}{\mu(U)} \int_X f(x)\overline{\chi(x, \omega)} d\mu(x), \quad \omega \in X^* \tag{9}$$

and it extends to the space $L^2(X)$ in the standard way.

Theorem 6. *If $f \in L^2(X)$, then $\widehat{f} \in L^2(X^*)$ and*

$$\frac{1}{\mu(U)} \int_X |f(x)|^2\, d\mu(x) = \frac{1}{\mu(U^*)} \int_{X^*} |\widehat{f}(\omega)|^2\, d\mu(\omega). \tag{10}$$

If, moreover, $f, g \in L^2(X)$, then we have

$$\frac{1}{\mu(U)} \int_X f(x)\overline{g(x)}\, d\mu(x) = \frac{1}{\mu(U^*)} \int_{X^*} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\, d\mu(\omega). \tag{11}$$

Proposition 7. *If $f \in L^2(X)$ and $\gamma \in H$, then*

$$\widehat{f(\cdot \oplus \gamma)}(\omega) = \widehat{f}(\omega)\chi(\gamma, \omega), \quad \omega \in X^*.$$

Let us introduce the following classes of step functions f given on X.

$$\mathcal{S}_n(X) := \{f : f(x) = f(x') \text{ for all } x, x' \in U_{n,k},\ k \in \mathbb{Z}_+\},$$
$$\mathcal{S}_n^{(k)}(X) := \{f \in \mathcal{S}_n(X) : \operatorname{supp} f \subset M^k(U)\}, \quad \mathcal{S}(X) := \bigcup_{n,k} \mathcal{S}_n^{(k)}(X).$$

The elements of the class $\mathcal{S}(X)$ are called the test functions. The classes $\mathcal{S}_n(X^*)$, $\mathcal{S}_n^{(k)}(X^*)$ and $\mathcal{S}(X^*)$ are defined similarly.

Theorem 8. *Let $f \in L^2(X)$. For $f \in \mathcal{S}_n(X)$, it is necessary and sufficient that* $\operatorname{supp} \widehat{f} \subset (M^*)^n(U^*)$. *In particular, for every integers n and k, there holds*

$$f \in \mathcal{S}_n^{(k)}(X) \ \Leftrightarrow\ \widehat{f} \in \mathcal{S}_k^{(n)}(X^*).$$

Proposition 9. *The set $\mathcal{S}(X)$ is dense in $L^2(X)$.*

Theorem 10. *For every $g \in L^2(X^*)$ there exists $f \in L^2(X)$ such that $\widehat{f} = g$. This function f is the inverse Fourier transform of g, which is defined on $L(X^*)$ by the formula*

$$f(x) = \frac{1}{\mu(U^*)} \int_{X^*} g(\omega)\chi(x, \omega)\, d\mu(\omega),$$

and extends to $L^2(X^*)$ in the same way as the Fourier transform.

Definition 11. The Walsh function is a function W_α, $\alpha \in \mathbb{Z}_+$, defined on X^* by the formula

$$W_\alpha(\omega) := \chi^*(\omega, \gamma_{[\alpha]}) = \chi(\gamma_{[\alpha]}, \omega);$$

a linear combination of the Walsh functions W_α, $\alpha < m^n$, i.e.

$$m_0(\omega) = \sum_{k=0}^{m^n-1} a_k W_k(\omega) \tag{12}$$

is called the Walsh polynomial of order n.

Proposition 12. *A function $w : X \to \mathbb{C}$ is a Walsh polynomial of order n if and only if it is an H-periodic function (i.e. $w(u \oplus h) = w(u)$ for all $u \in U$ and $h \in H$) that is constant on each set $U_{n,k}$, $0 \le k \le m^n - 1$.*

Theorem 13. *The system $\{W_\alpha/\sqrt{\mu U^*}\}_{\alpha=0}^{\infty}$ is orthonormal in $L^2(U^*)$.*

3. Refinable Functions

We are going to construct wavelet frames following the standard scheme associated with a multiresolution analysis generated by a refinable function.

Let m_0 be a Walsh polynomial of order n. Remind that, by Definition 2 and Theorem 1, for every $\omega \in X^*$ there is either a unique representation or a unique finite representation (which is always chosen) of the form

$$\omega = \sum_{j=-\infty}^{\infty} M^{*-j}\omega_j = \sum_{j=N}^{\infty} M^{*-j}\omega_j, \quad \omega_j \in D^*,$$

that we are writing as follows:

$$\omega = \omega_{-N}\dots\omega_0, \quad \omega_1\omega_2\dots$$

or

$$\omega = \omega_{-N}\dots\omega_0, \quad \omega_1\dots\omega_n$$

in the case of finite representation, and, due to Theorem 12, we have

$$m_0(\omega) = m_0(\mathbf{0}, \omega_1, \dots, \omega_n). \tag{13}$$

In what follows, we consider Walsh polynomials m_0 of order n such that $m_0(\mathbf{0}) = 1$ and denote the class of such polynomials by $\mathcal{M}_{\mathbf{0}}^{(\mathbf{n})}$.

Definition 14. A function $\varphi \in L^2(X)$ is called refinable if its Fourier transform satisfies the following *refinement equation:*

$$\widehat{\varphi}(\omega) = m_0((M^*)^{-1}\omega)\widehat{\varphi}((M^*)^{-1}\omega), \quad \omega \in X^*, \tag{14}$$

where $m_0 \in \mathcal{M}_{\mathbf{0}}^{(\mathbf{n})}$. The Walsh polynomial m_0 is called the mask of the refinable function φ.

Proposition 15. *If $m_0 \in \mathcal{M}_0^{(n)}$ satisfies the following condition*:

$$\sum_{s^* \in D^*} \left| m_0\big(\omega \oplus (M^*)^{-1}s^*\big)\right|^2 \le 1, \quad \forall\, \omega \in X^*, \tag{15}$$

then the function

$$g(\omega) := \prod_{j=1}^{\infty} m_0((M^*)^{-j}\omega), \quad \omega \in X^*, \tag{16}$$

belongs to $L^2(X^*) \cap \mathcal{S}_{n-1}(X^*)$ and a function φ such that $\widehat{\varphi} = Cg$ is a compactly supported refinable function with the mask m_0.

Proof. Setting

$$f_0 = \mathbf{1}_{U^*}, \quad f_k = \prod_{j=1}^{k} m_0((M^*)^{-j}\cdot)\mathbf{1}_{(M^*)^k(U^*)}, \quad k \in \mathbb{N}, \tag{17}$$

due to (5), for every $k \in \mathbb{N}$, we have

$$\begin{aligned}\int_{X^*} |f_{k+1}|^2\,d\mu &= \sum_{s^*\in D^*} \int_{(M^*)^k U^* + (M^*)^k s^*} |f_{k+1}|^2\,d\mu \\ &= \sum_{s^*\in D^*} \int_{(M^*)^k U^*} |f_{k+1}(\omega + (M^*)^k s^*)|^2\,d\mu(\omega) \\ &= \sum_{s^*\in D^*} \int_{(M^*)^k U^*} \left| \prod_{j=1}^{k+1} m_0\Big((M^*)^{-j}\omega + (M^*)^{k-j}s^*\Big)\right|^2 d\mu(\omega).\end{aligned}$$

Because of H^*-periodicity of m_0, (15) and the equality

$$(M^*)^{-j}\omega + (M^*)^{k-j}s^* = (M^*)^{-j}\omega \oplus (M^*)^{k-j}s^*, \quad \omega \in (M^*)^k U^*,$$

it follows that

$$\begin{aligned}&\int_{X^*} |f_{k+1}|^2\,d\mu \\ &= \int_{(M^*)^k U^*} \left| \prod_{j=1}^{k} m_0\Big((M^*)^{-j}\omega\Big)\right|^2 \sum_{s^*\in D^*} \left| m_0\Big((M^*)^{-k-1}\omega + (M^*)^{-1}s^*\Big)\right|^2 d\mu(\omega) \\ &\le \int_{(M^*)^k U^*} \left| \prod_{j=1}^{k} m_0\Big((M^*)^{-j}\omega\Big)\right|^2 d\mu(\omega) = \int_{X^*} |f_k|^2\,d\mu.\end{aligned}$$

Reiterating this $k+1$ times, we obtain

$$\int_{X^*} |f_{k+1}|^2\,d\mu \le \int_{X^*} |f_0|^2\,d\mu = \int_{U^*} d\mu = \mu(U^*).$$

Since $m_0(\mathbf{0}) = 1$, by Theorem 12, the polynomial m_0 is identically equal to 1 on the set $U^*_{n,0}$. Therefore, for each $\omega \in X^*$, the product (16) can contain only a finite number of factors different from 1. Hence, this product converges for any $\omega \in X^*$, and, due to Fatou's theorem,

$$\int_{X^*} |g|^2\,d\mu \le \sup_k \int_{X^*} |f_k|^2\,d\mu \le \mu(U^*).$$

It follows that $g \in L^2(X^*)$. Since, by Theorem 12, for every $j \ge 1$ the functions $m_0((M^*)^{-j}\cdot)$ are constant on the sets $U^*_{n-1,s}$, $s \in \mathbb{Z}_+$, we have $g \in \mathcal{S}_{n-1}(X^*)$. According to Theorems 10 and 8, if a function $\varphi \in L^2(X)$ is such that $\widehat{\varphi} = Cg$, then it is compactly supported and, obviously, relation (14) is satisfied, i.e. φ is refinable. $\square$

Thus, given a suitable Walsh polynomial $m_0 \in \mathcal{M}_0^{(n)}$, for any constant C we have a refinable function φ such that

$$\widehat{\varphi}(\omega) = C \prod_{j=1}^{\infty} m_0(M^{*-j}\omega), \quad C > 0, \tag{18}$$

and, because of (13), this equality takes the form

$$\widehat{\varphi}(\omega) = Cm_0(\mathbf{0}, \omega_0 \dots \omega_{n-1}) \dots, m_0(\mathbf{0}, \mathbf{0} \dots \mathbf{0}\omega_{-N}). \tag{19}$$

Obviously, for any refinable function φ with a mask m_0, its Fourier transform is given by (18). Since $m_0 \in \mathcal{S}_n$ (because of Theorem 12), it follows from (19) that $\widehat{\varphi} \in \mathcal{S}_{n-1}(X^*)$ and, by Theorem 8, the function φ is compactly supported. However, we want to have refinable functions that are the step functions, i.e. we are interested in refinable functions from the space $\mathcal{S}(X)$.

Definition 16. Given $r \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $m_0 \in \mathcal{M}_0^{(n)}$, $\sigma_r = \sigma_r(m_0)$ is the set of vectors

$$(\xi_0, \xi_1, \dots, \xi_r), \quad \xi_j \in D^*, \quad \xi_0 \neq \mathbf{0},$$

such that

$$m_0(\mathbf{0}, \xi_{1-n+l} \dots \xi_l) \neq 0, \quad \forall\, l \in \{0, 1, \dots, r\}, \tag{20}$$

where $\xi_j = \mathbf{0}$ whenever $j < 0$;

$\sigma_\infty = \sigma_\infty(m_0)$ is the set of sequences

$$(\xi_0, \xi_1, \xi_2, \dots), \quad \xi_j \in D^*, \quad \xi_0 \neq \mathbf{0},$$

such that (20) holds for all $r \geq 0$.

Note the following obvious properties of these new notions. If $\sigma_r = \emptyset$, then $\sigma_{r+1} = \emptyset$, and if $\sigma_\infty \neq \emptyset$, then $\sigma_r \neq \emptyset$ for all r.

Lemma 17. *Suppose that $m_0 \in \mathcal{M}_0^{(n)}$ is the mask of a refinable test function φ. Then $\sigma_\infty = \emptyset$.*

Proof. Assume that σ_∞ contains a sequence $(\xi_0, \xi_1, \dots)$ with $\xi_0 \neq 0$. Then, by (19),

$$\begin{aligned}
&\widehat{\varphi}(\xi_0 \dots \xi_N, \xi_{N+1} \dots \xi_{N+n-1}) \\
&\quad = m_0(\mathbf{0}, \xi_N \dots \xi_{N+n-1}) m_0(\mathbf{0}, \xi_{N-1} \dots \xi_{N+n-2}) \dots m_0(\mathbf{0}, \mathbf{0} \dots \mathbf{0}\xi_0) \neq 0
\end{aligned}$$

for every $N \in \mathbb{N}$. Thus, the function $\widehat{\varphi}$ is not compactly supported, which contradicts our assumption that φ is a test function, because of Theorem 8. □

Lemma 18. *Suppose that $m_0 \in \mathcal{M}_0^{(n)}$ and $r \in \mathbb{Z}_+$. If $\sigma_r = \emptyset$, then the function*

$$g(\omega) := \prod_{k=1}^{\infty} m_0(M^{*-k}\omega)$$

belongs to the space $\mathcal{S}_{n-1}^{(r-n+1)}(X^*)$.

Proof. First of all we note that $g(\omega)$ is finite for every ω because, by (13), we have $m_0(M^{*-k}\omega) = m_0(\mathbf{0}) = 1$ whenever k is large enough. Since $m_0 \in \mathcal{S}_n(X^*)$, obviously, g belongs to the space $\mathcal{S}_{n-1}(X^*)$. Let us check that $\operatorname{supp} g \subset M^{*(r-n+1)}(U^*)$. Assume that there exists $\omega \in X^*$ such that $\omega \notin M^{*(r-n+1)}(U^*)$ and $g(\omega) \neq 0$. If now $\omega = M^{*(r-n+l+1)}(\xi_0, \xi_1, \xi_2, \ldots)$, where $l \in \mathbb{Z}_+$, $\xi_j \in D^*$ and $\xi_0 \neq \mathbf{0}$, then $m_0(M^{*-k}\omega) \neq 0$ for all $k \in \mathbb{N}$. Due to (13), for every positive integer $k \leq r+1$, we have

$$\begin{aligned} m_0((M^*)^{-k-l}\omega) &= m_0\big((M^*)^{r+1-n-k}(\xi_0, \xi_1, \xi_2, \ldots)\big) \\ &= m_0(\mathbf{0}, \xi_{r-n-k+2} \ldots \xi_{r-k+1}) \neq 0, \end{aligned}$$

where $\xi_j = \mathbf{0}$ whenever $j < 0$. Hence the vector $(\xi_0, \xi_1, \ldots, \xi_r)$ is in σ_r, which contradicts our assumption that $\sigma_r = \emptyset$. □

Theorem 19. *Let $m_0 \in \mathcal{M}_0^{(n)}$, $n \in \mathbb{N}$, and let $r = m^{n-1} - 1$. For the function m_0 to be the mask of a refinable test function it is necessary and sufficient that $\sigma_r = \emptyset$.*

Proof. The sufficiency follows from Lemma 18, Theorems 8 and 10.

To prove the necessity, we assume that m_0 is the mask of a refinable test function φ. By Theorem 8, it follows that $\widehat{\varphi}$ is also a test function. Assume that $\sigma_r \neq \emptyset$, which means the existence of a vector $(\xi_0, \ldots, \xi_r) \in \sigma_r$.

If $\xi_{l+1} = \mathbf{0}, \ldots, \xi_{l+n-1} = \mathbf{0}$ for some $l \leq r - n + 1$, then

$$(\xi_0, \ldots, \xi_l, \mathbf{0}, \mathbf{0}, \ldots) \in \sigma_\infty,$$

which contradicts Lemma 17.

Since there are at most r vectors $(\xi_{k+1}, \ldots, \xi_{k+n-1})$, $\xi_j \in D^*$, different from $(\mathbf{0}, \ldots, \mathbf{0})$, there exist k_1 and k_2, $-n+2 \leq k_1 < k_2 \leq r - n + 2$, such that

$$\xi_{k_1+j} = \xi_{k_2+j}, \quad j = 0, \ldots, n-2. \tag{21}$$

If $k_2 - k_1 > n - 1$, then the vector $(\xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_2}, \ldots, \xi_{k_2+n-2})$ is a part of

$$(\xi_{-n+2}, \ldots, \xi_{-1}, \xi_0, \ldots, \xi_r).$$

Hence, taking into account (21), we have

$$(\xi_0, \ldots, \xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots) \in \sigma_\infty \tag{22}$$

whenever $k_1 \geq 0$, and

$$(\xi_0, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots, \xi_{k_2-1}, \ldots) \in \sigma_\infty \tag{23}$$

whenever $k_1 < 0$.

Let now $k_2 - k_1 \leq n-1$, and let L be the maximal integer such that $L(k_2 - k_1) \leq n - 1$. Since the vector

$$(\xi_{k_1}, \ldots, \xi_{k_2-1}, \xi_{k_1}, \ldots, \xi_{k_2-1}, \ldots, \xi_{k_1}, \ldots, \xi_{k_2-1}),$$

where $\xi_{k_1}, \dots, \xi_{k_2-1}$ is taken $L+1$ times, is a part of

$$(\xi_{-n+2}, \dots, \xi_{-1}, \xi_0, \dots, \xi_r),$$

again (22) or (23) holds, which contradicts Lemma 17. □

Due to Theorem 19, to construct a refinable test function one has to choose $m_0 \in \mathcal{M}_0^{(n)}$ such that for every chain of digits

$$\xi_0, \xi_1, \dots, \xi_r, \quad r = m^{n-1} - 1, \quad \xi_j \in D^*, \quad \xi_0 \neq \mathbf{0}$$

there exists $l \le r$ such that

$$m_0(\mathbf{0}, \xi_{1-n+l} \dots \xi_l) = 0,$$

where $\xi_j = \mathbf{0}$ whenever $j < 0$. To see all possibilities choosing zeros of m_0, it is convenient to present the vectors $(\xi_{1-n+l}, \dots, \xi_l)$ as tree vertices (see Fig. 2 for the case $m = n = 3$, where for simplicity we write $k_1k_2k_3$ instead of the vector $(s^*_{k_1}, s^*_{k_2}, s^*_{k_3})$. We set the null vector to the root, and organize each branch as follows:

$$(0, \dots, 0) \to \dots \to (\xi_{1-n+l}, \dots, \xi_l) \to (\xi_{2-n+l}, \dots, \xi_{l+1}) \to \cdots.$$

Consider the case $m = n = 3$ in more detail. Note that four first columns of the tree contain all elements corresponding to different vectors (ξ_1, ξ_2, ξ_3), $\xi_j \in D^*$, so m_0 is defined by setting all the values $m_0(\mathbf{0}, \xi_1\xi_2\xi_3)$. We need to provide at least one zero of m_0 on every branch of the tree. The simplest way is to choose zeros in the elements of the second column. However, if we want φ to have smaller size of the steps, it's worth to keep non-zero elements as long as possible. Each of the

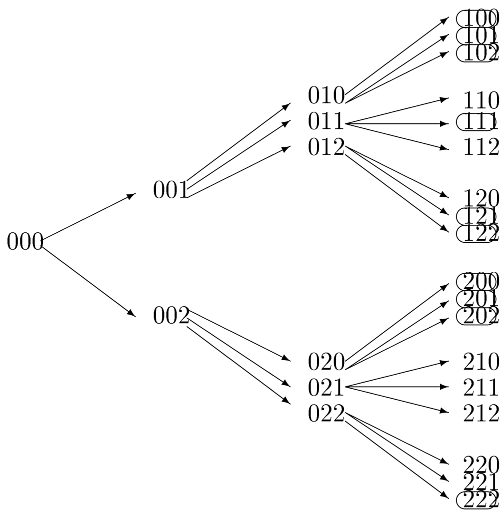


Fig. 2. The scheme for choosing zeros in the case $m = n = 3$.

zeros in the second column can be replaced by three zeros from the third column. If there are no zeros in first three columns, we can vanish m_0 in some elements of the fourth column. It is clear from the proof of Theorem 19, that m_0 must be vanished in the points $\mathbf{0}, \xi_1\xi_2\xi_3$ corresponding to the elements

$$100, 101, 111, 200, 202, 222.$$

Taking into account that the elements $100, 101, 102$ follow again after 110 and 210 and the elements $200, 201, 202$ follow again after 120 and 220, we see that it is not necessary to vanish m_0 in the points, corresponding to 110, 210, 120 and 220. Next, since the elements $120, 121, 122$ follow again after 112 and 212, the elements $110, 111, 112$ follow again after 211, and the elements $210, 211, 212$ follow again after 221, we see that it suffices to vanish m_0 only in the points, corresponding to the circled elements.

4. Tight Wavelet Frames on X

Let $\psi^{(1)}, \dots, \psi^{(q)} \in L^2(X)$, where $q \geq m-1$. The set of functions

$$\{\psi^{(\nu)}_{j,k} : 1 \leq \nu \leq q, j \in \mathbb{Z}, k \in \mathbb{Z}_+\},$$

where

$$\psi^{(\nu)}_{j,k}(x) = m^{j/2}\psi^{(\nu)}(M^j x \oplus \gamma_{[k]}), \quad x \in X,$$

is called the *wavelet system*, generated by the *wavelet functions* $\psi^{(\nu)}$.

A wavelet system is called a *tight wavelet frame* in $L^2(X)$ if

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} \sum_{\nu=1}^{q} |\langle f, \psi^{(\nu)}_{j,k}\rangle|^2 = B\|f\|^2, \quad B > 0, \tag{24}$$

for every $f \in L^2(X)$. If $B = 1$, then the system is said to be a *Parseval frame*.

The above concept of wavelet tight/Parseval frame is a special case of the well-known notion of a frame in a Hilbert space, and the general frame theory is deeply investigated (see, e.g., [16, Sec. 1.2]). In particular, it is known that a wavelet system $\{\psi^{(\nu)}_{j,k}\}_{j,k,\nu}$ is a Parseval frame if and only if for every $f \in L^2(X)$ there holds

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} \sum_{\nu=1}^{q} \langle f\, \psi^{(\nu)}_{j,k}\rangle \psi^{(\nu)}_{j,k}.$$

Now we describe a method for constructing tight wavelet frames in $L^2(X)$ based on a suitable Walsh polynomial m_0 given by (12).

Lemma 20. *Let $m_0 \in \mathcal{M}^{(n)}_0$ satisfy* (15) *and $q \geq m-1$ ($q > m-1$ in the case of the strict inequality in relations* (15)). *Then there exist Walsh polynomials $m_1, \dots, m_q$*

of order n satisfying the following condition:

$$\sum_{\nu=0}^{q} m_\nu((M^*)^{-1}(\xi \oplus s_k^*))\overline{m_\nu((M^*)^{-1}(\xi \oplus s_{k'}^*))} = \delta_{k,k'}, \quad \forall\, \xi \in X^*. \tag{25}$$

Proof. First of all we note that (15) may be rewritten in the form

$$\sum_{k=0}^{m-1} \Big| m_0\big(\mathbf{0}, s_k^*\xi_2 \dots \xi_n\big)\Big|^2 \le 1, \quad \forall\, (\xi_2, \dots, \xi_n), \quad \xi_j \in D^*. \tag{26}$$

For every fixed $(\xi_2, \dots, \xi_n)$, we set $b_{0k} = m_0(\mathbf{0}, s_k^*\xi_2 \dots \xi_n)$, $k = 0, \dots, m-1$, choose numbers $b_{0m}, \dots, b_{0q}$, such that $|b_{00}|^2 + \dots + |b_{0q}|^2 = 1$. Note that one can take $q = m-1$ only in the case $|b_{00}|^2 + \dots + |b_{0,m-1}|^2 = 1$. Using the Householder transform, we find numbers $b_{\nu k}$, $\nu = 1, \dots, q$, $k = 0, \dots, q$, such that the matrix $B = \{b_{\nu k}\}_{\nu,k=0}^{q}$ is unitary, and set

$$m_\nu(\mathbf{0}, s_k^*\xi_2 \dots \xi_n) := b_{\nu k}, \quad k = 0, \dots, m-1.$$

Since each set $U_{n,l}^*$, $0 \le l \le m^n - 1$ contains one and only one point $\mathbf{0}, s_k^*\xi_2 \dots \xi_n$, we extend m_ν onto every $U_{n,l}^*$, and then extend it H^*-periodically onto the entire space X^*. Thus we defined Walsh polynomials m_ν such that

$$\sum_{\nu=0}^{q} m_\nu\big(\mathbf{0}, s_k^*, \xi_2 \dots \xi_n\big)\overline{m_\nu\big(\mathbf{0}, s_{k'}^*\xi_2 \dots \xi_n\big)} = \delta_{k,k'}, \quad \forall\, (\xi_2, \dots, \xi_n), \quad \xi_j \in D^*,$$

that is equivalent to (25). □

Let now $m_0 \in \mathcal{M}_0^{(n)}$ satisfy (15) and $\sigma_r(m_0) = \emptyset$, $r = m^{n-1} - 1$. Because of Theorem 19, the function m_0 is the mask of a compactly supported refinable step function φ, i.e. $\varphi \in \mathcal{S}(X)$.

Definition 21. Let $m_0 \in \mathcal{M}_0^{(n)}$ be the mask of a refinable step function φ, and let m_ν, $\nu = 1, \dots, q$, be the Walsh polynomials from Lemma 20. The functions $\psi^{(1)}, \dots, \psi^{(q)}$, defined by

$$\widehat{\psi}^{(\nu)}(\omega) = m_\nu((M^*)^{-1}\omega)\widehat{\varphi}((M^*)^{-1}\omega), \quad \omega \in X^*, \quad \nu = 1, \dots, q, \tag{27}$$

are called the wavelet functions associated with φ.

Since $\varphi \in \mathcal{S}(X)$, obviously, the functions $\psi^{(\nu)}$ also belong to $\mathcal{S}(X)$.

Theorem 22. *Let $m_0 \in \mathcal{M}_0^{(n)}$ be the mask of a refinable step function φ, and let $\psi^{(1)}, \dots, \psi^{(q)}$, $q \ge m-1$, be the wavelet functions associated with φ. Then the wavelet system generated by $\psi^{(1)}, \dots, \psi^{(q)}$ is a tight frame in $L^2(X)$, and this frame is a Parseval frame whenever $\widehat{\varphi}(\mathbf{0}) = \mathbf{1}/\sqrt{\mu(\mathbf{U})}$.*

To prove this theorem we need a number of auxiliary results.

Lemma 23. *Let $f \in L^2(X)$, $\psi \in \mathcal{S}(X)$. Then the function*

$$[\widehat{f}, \widehat{\psi}](\omega) := \sum_{h \in H^*} \widehat{f}(\omega + h)\overline{\widehat{\psi}(\omega + h)}, \quad \omega \in U^*,$$

belongs to $L^2(U^*)$, and

$$\sum_{k \in \mathbb{Z}_+} \left|\langle f, \psi(\cdot \oplus \gamma_{[k]})\rangle\right|^2 = \frac{\mu^2(U)}{\mu(U^*)} \int_{U^*} \left|[\widehat{f}, \widehat{\psi}]\right|^2 d\mu. \tag{28}$$

Proof. By the Cauchy inequality, we have

$$\int_{U^*} \left|\sum_{h \in H^*} \widehat{f}(\omega + h)\overline{\widehat{\psi}(\omega + h)}\right|^2 d\mu(\omega)$$
$$\le \int_{U^*} \sum_{h \in H^*} |\widehat{f}(\omega + h)|^2 \sum_{g \in H^*} |\widehat{\psi}(\omega + g)|^2 \, d\mu(\omega),$$

where the sum with respect to g is finite. It follows from Theorem 6 that

$$\int_{U^*} \left|\sum_{h \in H^*} \widehat{f}(\omega + h)\overline{\widehat{\psi}(\omega + h)}\right|^2 d\mu(\omega)$$
$$\le C_\psi \sum_{h \in H^*} \int_{U^* + h} |\widehat{f}(\omega)|^2 \, d\mu(\omega) \le C_\psi \frac{\mu U^*}{\mu U} \|f\|_2^2,$$

which yields $[\widehat{f}, \widehat{\psi}] \in L^2(U^*)$.

Again using Theorem 6 and Proposition 7, taking into account that $\chi(h, \gamma_{[k]}) = 1$ for all $h \in H^*$ and $\omega + h = \omega \oplus h$ whenever $\omega \in U^*, h \in H^*$, we get

$$\begin{aligned}
\frac{1}{\mu(U)}\langle f, \psi(\cdot \oplus \gamma_{[k]})\rangle &= \frac{1}{\mu(U^*)} \int_{X^*} \widehat{f}(\omega)\overline{\widehat{\psi}(\omega)\chi(\omega, \gamma_{[k]})} \, d\mu(\omega) \\
&= \frac{1}{\mu(U^*)} \sum_{h \in H^*} \int_{U^* + h} \widehat{f}(\omega)\overline{\widehat{\psi}(\omega)\chi(\omega, \gamma_{[k]})} \, d\mu(\omega) \\
&= \frac{1}{\mu(U^*)} \int_{U^*} \sum_{h^* \in H^*} \widehat{f}(\omega + h)\overline{\widehat{\psi}(\omega + h)\chi(\omega + h, \gamma_{[k]})} \, d\mu(\omega) \\
&= \frac{1}{\mu(U^*)} \int_{U^*} [\widehat{f}, \widehat{\psi}](\omega)\overline{W_k(\omega)} \, d\mu(\omega),
\end{aligned}$$

i.e. the product $\sqrt{\mu U^*}(\mu(U))^{-1}\langle f, \psi(\cdot \oplus \gamma_{[k]})\rangle$ coincides with the kth Fourier coefficient of the function $[\widehat{f}, \widehat{\psi}]$ with respect to the orthonormal system $\{W_k/\sqrt{\mu(U^*)}\}_k$. Using Parseval's equality, we obtain (28). $\square$

Lemma 24. *If $\varphi \in \mathcal{S}(X)$ and $f \in L^2(X)$, then*

$$\sum_{k\in\mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 \le C_\varphi \|f\|_2^2 \|\varphi\|_2^2, \quad \forall\, j \in \mathbb{Z}. \tag{29}$$

Proof. First we will prove that the system $\{\varphi(\cdot \oplus h)\}_{h\in H}$ is Bessel. Let

$$c = \{c_h\}_{h\in H} : \sum_{h\in H} |c_h|^2 < \infty.$$

Using that $x + \gamma = x \oplus \gamma$ whenever $x \in U$, $\gamma \in H$, we have

$$\begin{aligned}\int_X \left|\sum_{h\in H} c_h\, \varphi(x\oplus h)\right|^2 d\mu(x) &= \sum_{\gamma\in H}\int_{U+\gamma}\left|\sum_{h\in H} c_h\, \varphi(x\oplus h)\right|^2 d\mu(x)\\ &= \sum_{\gamma\in H}\int_U \left|\sum_{h\in H} c_h\, \varphi((x+\gamma)\oplus h)\right|^2 d\mu(x)\\ &= \sum_{\gamma\in H}\int_U \left|\sum_{h\in H} c_h\, \varphi(x\oplus(\gamma\oplus h))\right|^2 d\mu(x).\end{aligned}$$

For every $\gamma \in H$, setting $\Omega_\gamma = \{h \in H :\ \exists x \in U : \text{s.t.} : \varphi(x + (\gamma\oplus h)) \ne 0\}$ and using Cauchy's inequality, we obtain

$$\begin{aligned}\int_U \left|\sum_{h\in H} c_h\, \varphi(x\oplus(\gamma\oplus h))\right|^2 d\mu(x) &= \int_U \left|\sum_{h\in \Omega_\gamma} c_h\, \varphi(x\oplus(\gamma\oplus h))\right|^2 d\mu(x)\\ &\le \sum_{h\in\Omega_\gamma} |c_h|^2 \sum_{h'\in H}\int_U |\varphi(x\oplus(\gamma\oplus h'))|^2\, d\mu(x).\end{aligned}$$

Since $\{\gamma\oplus h'\}_{h'\in H} = \{h''\}_{h''\in H}$ for any $\gamma \in H$, there holds

$$\begin{aligned}&\sum_{h'\in H}\int_U |\varphi(x\oplus(\gamma\oplus h'))|^2\, d\mu(x)\\ &\quad= \sum_{h'\in H}\int_U |\varphi(x+(\gamma\oplus h'))|^2\, d\mu(x)\\ &\quad= \sum_{h''\in H}\int_U |\varphi(x+h'')|^2\, d\mu(x) = \int_X |\varphi(x)|^2\, d\mu(x) = \|\,\varphi\,\|_2^2.\end{aligned}$$

Obviously, Ω_γ is contained in a neighborhood of γ depending only on the support of φ, which yields

$$\sum_{\gamma\in H}\sum_{h\in\Omega_\gamma} |c_h|^2 \le C_\varphi \|c\|_{\ell_2}^2.$$

Combining this with the previous estimate gives

$$\int_X \left| \sum_{h\in H} c_h\, \varphi(x\oplus h) \right|^2 d\mu(x) \le C_\varphi \|c\|_{\ell_2}^2 \|\,\varphi\,\|_2^2,$$

which means that the system $\{\varphi(\cdot\oplus h)\}_{h\in H}$ is Bessel. This yields (see, e.g., [21, Theorem 1.1.2 and Remark 1.1.3]) that

$$\sum_{k\in\mathbb{Z}_+} |\langle f, \varphi(\cdot\oplus\gamma_{[k]})\rangle|^2 \le C_\varphi \|f\|_2^2 \|\varphi\|_2^2.$$

Thus lemma is proved for $j=0$. To verify (29) for arbitrary j it remains to note that $\langle f, \varphi_{j,k}\rangle = \langle m^{-j/2} f(M^{-j}\cdot), \varphi(\cdot\oplus\gamma_{[k]})\rangle$ and $\|m^{-j/2} f(M^{-j}\cdot)\|_2 = \|f\|_2$. □

Lemma 25. *If $\varphi\in\mathcal{S}(X)$ and $f\in L^2(X)$, then*

$$\lim_{j\to+\infty} \sum_{k\in\mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 = \mu(U)|\widehat{\varphi}(0)|^2 \|f\|_2^2. \tag{30}$$

Proof. First we consider the case where $f\in\mathcal{S}(X)$. Using Lemma 23, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 &= \sum_{k\in\mathbb{Z}_+} |\langle m^{-j/2} f(M^{-j}\cdot), \varphi(\cdot\oplus\gamma_{[k]})\rangle|^2 \\ &= m^j \frac{\mu^2(U)}{\mu(U^*)} \int_{U^*} \left|[\widehat{f}(M^{*j}\cdot), \widehat{\varphi}](\omega)\right|^2 d\mu(\omega). \end{aligned}\tag{31}$$

If $\omega\in U^*$, $h', \widetilde{h}\in H^*$ and $h'\oplus\widetilde{h} = \mathbf{0}$ then

$$\begin{aligned}[\widehat{f}(M^{*j}\cdot), \widehat{\varphi}] &= \sum_{h\in H^*} \widehat{f}((M^*)^j(\omega+h))\overline{\widehat{\varphi}(\omega+h)} \\ &= \sum_{h\in H^*} \widehat{f}((M^*)^j(\omega\oplus h'\oplus\widetilde{h}\oplus h))\overline{\widehat{\varphi}(\omega\oplus h'\oplus\widetilde{h}\oplus h)} \\ &= \sum_{h\in H^*} \widehat{f}((M^*)^j((\omega+h')\oplus h))\overline{\widehat{\varphi}((\omega+h')\oplus h)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{U^*} \left|[\widehat{f}(M^{*j}\cdot), \widehat{\varphi}](\omega)\right|^2 d\mu(\omega) \\ &\quad = \int_{U^*} \sum_{h\in H^*} \widehat{f}((M^*)^j(\omega+h))\overline{\widehat{\varphi}(\omega+h)} \sum_{h'\in H^*} \overline{\widehat{f}((M^*)^j(\omega+h'))\overline{\widehat{\varphi}(\omega+h')}}\, d\mu(\omega)\end{aligned}$$

$$= \sum_{h' \in H^*} \int_{U^*} \overline{\widehat{f}((M^*)^j(\omega + h'))\overline{\widehat{\varphi}(\omega + h')}}$$

$$\times \sum_{h \in H^*} \widehat{f}((M^*)^j((\omega + h') \oplus h))\overline{\widehat{\varphi}((\omega + h') \oplus h)}\, d\mu(\omega)$$

$$= \int_{X^*} \overline{\widehat{f}((M^*)^j(\omega))\overline{\widehat{\varphi}(\omega)}} \sum_{h \in H^*} \widehat{f}((M^*)^j(\omega \oplus h))\overline{\widehat{\varphi}(\omega \oplus h)}\, d\mu(\omega)$$

$$= m^{-j} \int_{X^*} \overline{\widehat{f}(\omega)\overline{\widehat{\varphi}((M^*)^{-j}\omega)}} \sum_{h \in H^*} \widehat{f}(\omega \oplus ((M^*)^j h))\overline{\widehat{\varphi}((M^*)^{-j}\omega \oplus h)}\, d\mu(\omega).$$

Obviously, there exists j_0 such that if $j \geq j_0$, $h' \in H^*$ and $h' \neq \mathbf{0}$, then

$$\widehat{f}(\omega)\widehat{f}(\omega \oplus (M^*)^j h') = 0, \quad \forall\, \omega \in X^*.$$

Hence, there holds

$$\int_{U^*} \left|[\widehat{f}(M^{*j}\cdot), \widehat{\varphi}](\omega)\right|^2 d\mu(\omega)$$
$$= m^{-j} \int_{X^*} |\widehat{f}(\omega)|^2 |\widehat{\varphi}((M^*)^{-j}\omega)|^2\, d\mu(\omega), \quad \forall\, j \geq j_0.$$

Together with (31), this yields

$$\sum_{k \in \mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 = \frac{\mu^2(U)}{\mu(U^*)} \int_{X^*} |\widehat{f}(\omega)|^2 |\widehat{\varphi}((M^*)^{-j}\omega)|^2\, d\mu(\omega), \quad \forall\, j \geq j_0. \tag{32}$$

Since $\lim_{j \to +\infty} \widehat{\varphi}((M^*)^{-j}\omega) = \widehat{\varphi}(0)$ for every $\omega \in X^*$, by Lebesgue's theorem and Theorem 6, the equality (30) follows from (32) by passing to the limit as $j \to +\infty$.

Let now f be an arbitrary function from $L^2(X)$. Due to Proposition 9, for every $\varepsilon > 0$ there exists $\widetilde{f} \in S(X)$ such that $\|f - \widetilde{f}\|_2 < \varepsilon$. Since (30) is proved already for the function $\widetilde{f}$, using Lemma 24 and the triangle inequality, we obtain (30) for f. □

Lemma 26. *If $f \in L^2(X)$ and $\varphi \in \mathcal{S}(X)$, then*

$$\lim_{j \to -\infty} \sum_{k \in \mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 = 0. \tag{33}$$

Proof. Repeating the final arguments of the previous proof, we see that it is sufficient to consider the case where $f \in \mathcal{S}(X)$. Thus we can assume that $\operatorname{supp} f \subset M^N(U)$ for some positive N. Suppose also that $j \leq -N$. Then, taking into account that $x + h = x \oplus h$ whenever $x \in M^{N+j}(U)$ and $h \in H$, we have

$$\sum_{k\in\mathbb{Z}_+}|\langle f,\varphi_{j,k}\rangle|^2 \le m^j\sum_{k\in\mathbb{Z}_+}\left(\int_{M^N(U)}|f(x)||\varphi(M^jx\oplus\gamma_{[k]})|\,d\mu(x)\right)^2$$

$$\le m^j\|f\|_2^2\sum_{k\in\mathbb{Z}_+}\int_{M^N(U)}|\varphi(M^jx\oplus\gamma_{[k]})|^2\,d\mu(x)$$

$$=\|f\|_2^2\sum_{k\in\mathbb{Z}_+}\int_{M^{N+j}(U)}|\varphi(x+\gamma_{[k]})|^2\,d\mu(x).$$

Since φ is compactly supported and U is bounded, there exists N' such that $\varphi(x+\gamma_{[k]})=0$ whenever $k>N'$ and $x\in M^{N+j}(U)$. This yields

$$\sum_{k\in\mathbb{Z}_+}|\langle f,\varphi_{j,k}\rangle|^2\le\|f\|_2^2\int_{S(N,j)}|\varphi(x)|^2\,d\mu(x), \tag{34}$$

where

$$S(N,j):=\bigcup_{0\le k\le N'}(M^{N+j}(U)+\gamma_{[k]}).$$

Obviously,

$$\lim_{j\to-\infty}\mathbf{1}_{S(N,j)}(x)=0\quad\text{for every } x\in X\setminus H.$$

It follows from Lebesgue's theorem that

$$\lim_{j\to-\infty}\int_X\mathbf{1}_{S(N,j)}|\varphi(x)|^2\,d\mu(x)=0,\quad\text{i.e.}\ \lim_{j\to-\infty}\int_{S(N,j)}|\varphi(x)|^2\,d\mu(x)=0.$$

Together with (34), this implies (33). □

Proof of Theorem 22. Let us prove that if $f\in L^2(X)$ and $j,j'\in\mathbb{Z}$, $j<j'$, then

$$\sum_{k\in\mathbb{Z}_+}|\langle f,\varphi_{j,k}\rangle|^2+\sum_{i=j}^{j'-1}\sum_{\nu=1}^{q}\sum_{k\in\mathbb{Z}_+}|\langle f,\psi_{i,k}^{(\nu)}\rangle|^2=\sum_{k\in\mathbb{Z}_+}|\langle f,\varphi_{j',k}\rangle|^2. \tag{35}$$

Obviously, it suffices to verify that (35) holds for $j'=j+1$, i.e. we need to prove that

$$\sum_{k\in\mathbb{Z}_+}|\langle f,\varphi_{j,k}\rangle|^2+\sum_{\nu=1}^{q}\sum_{k\in\mathbb{Z}_+}|\langle f,\psi_{j,k}^{(\nu)}\rangle|^2=\sum_{k\in\mathbb{Z}_+}|\langle f,\varphi_{j+1,k}\rangle|^2. \tag{36}$$

Remind that the functions $\psi^{(\nu)}$ are defined by (27), where m_ν are Walsh polynomials satisfying (25). Using Lemma 23, (14) and (27), taking into account H^*-periodicity of the functions $m_0, m_1, \dots, m_q$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 + \sum_{\nu=1}^{q}\sum_{k\in\mathbb{Z}_+} |\langle f, \psi^{(\nu)}_{j,k}\rangle|^2 \\
&\quad = \sum_{k\in\mathbb{Z}_+} |\langle m^{-j/2} f(M^{-j}\cdot), \varphi(\cdot\oplus\gamma_{[k]})\rangle|^2 \\
&\qquad + \sum_{\nu=1}^{q}\sum_{k\in\mathbb{Z}_+} |\langle m^{-j/2} f(M^{-j}\cdot), \psi^{(\nu)}(\cdot\oplus\gamma_{[k]})\rangle|^2 \\
&\quad = m^j \frac{\mu^2(U)}{\mu(U^*)} \int_{U^*} \left|[\widehat{f}(M^{*j}\cdot), \widehat{\varphi}](\xi)\right|^2 d\mu(\xi) \\
&\qquad + m^j \frac{\mu^2(U)}{\mu(U^*)} \sum_{\nu=1}^{q} \int_{U^*} \left|[\widehat{f}(M^{*j}\cdot), \widehat{\psi^{(\nu)}}](\xi)\right|^2 d\mu(\xi) \\
&\quad = m^j \frac{\mu^2(U)}{\mu(U^*)} \sum_{\nu=0}^{q} \int_{U^*} \sum_{h\in H^*} \widehat{f}\left(M^{*j}(\xi+h)\right) \\
&\qquad \times \overline{m_\nu\left(M^{*-1}(\xi+h)\right)\widehat{\varphi}\left(M^{*-1}(\xi+h)\right)} \\
&\qquad \cdot \overline{\sum_{h'\in H^*} \widehat{f}\left(M^{*j}(\xi+h')\right)\overline{m_\nu\left(M^{*-1}(\xi+h')\right)\widehat{\varphi}\left(M^{*-1}(\xi+h')\right)}}\, d\mu(\xi) \\
&\quad = m^j \frac{\mu^2(U)}{\mu(U^*)} \sum_{\nu=0}^{q} \int_{U^*} \left(\sum_{s\in D^*} [\widehat{f}(M^{*(j+1)}\cdot), \widehat{\varphi}](M^{*-1}(\xi+s))\overline{m_\nu\left(M^{*-1}(\xi+s)\right)}\right) \\
&\qquad \cdot \overline{\left(\sum_{s'\in D^*} [\widehat{f}(M^{*(j+1)}\cdot), \widehat{\varphi}](M^{*-1}(\xi+s'))\overline{m_\nu\left(M^{*-1}(\xi+s')\right)}\right)}\, d\mu(\xi). \qquad (37)
\end{aligned}$$

Since $\xi + s = \xi \oplus s$ whenever $\xi \in U^*, s \in D^*$, due to (25), there holds

$$\sum_{\nu=0}^{q} m_\nu\left(M^{*-1}(\xi+s)\right)\overline{m_\nu\left(M^{*-1}(\xi+s')\right)} = \delta_{ss'}, \quad s, s' \in D^*,$$

and hence the right-hand side of (37) can be reduced to

$$m^j \frac{\mu^2(U)}{\mu(U^*)} \int_{U^*} \sum_{s\in D^*} \left|[\widehat{f}(M^{*(j+1)}\cdot), \widehat{\varphi}](M^{*-1}(\xi+s))\right|^2 d\mu(\xi).$$

Using the equality $U^* = \cup_{s\in D^*} M^{*-1}(U^* + s)$, we obtain

$$\sum_{k\in\mathbb{Z}_+} |\langle f, \varphi_{j,k}\rangle|^2 + \sum_{\nu=1}^{q} \sum_{k\in\mathbb{Z}_+} |\langle f, \psi_{j,k}^{(\nu)}\rangle|^2$$

$$= m^{j+1} \frac{\mu^2(U)}{\mu(U^*)} \sum_{s\in D^*} \int_{M^{*-1}(U^*+s)} \left|[\widehat{f}(M^{*(j+1)}\cdot), \widehat{\varphi}](\xi)\right|^2 d\mu(\xi)$$

$$= m^{j+1} \frac{\mu^2(U)}{\mu(U^*)} \int_{U^*} \left|[\widehat{f}(M^{*j+1}\cdot), \widehat{\varphi}](\xi)\right|^2 d\mu(\xi).$$

To prove (36), it remains to use Lemma 23 once more. Hence the equality (35) holds true as well.

Using Lemmas 25, 26 and passing to the limit in (35) as $j \to -\infty$ and $j' \to +\infty$, we obtain (24) with $B = \mu(U)|\widehat{\varphi}(0)|^2$, which was to be proved. □

Example 1 (Haar system). Set $\varphi = \frac{1}{\sqrt{\mu U}} \mathbf{1}_U$. If $\omega \in X^*$ and $\omega = h \oplus \omega'$, where $h \in H^*$, $\omega' \in U^*$, then, due to Lemma 4, we have

$$\widehat{\varphi}(\omega) = \frac{1}{(\mu U)^{3/2}} \int_U \overline{\chi(x,\omega)}\, d\mu(x)$$

$$= \frac{1}{(\mu U)^{3/2}} \int_U \overline{\chi(x,h)}\, d\mu(x) = \frac{1}{\sqrt{\mu U}} \mathbf{1}_{U^*}$$

and $\widehat{\varphi}(\mathbf{0}) = \frac{1}{\sqrt{\mu U}}$. Obviously, the function φ is refinable with the mask $m_0 \in \mathcal{M}_0^{(1)}$ that coincides with the function $\mathbf{1}_{M^{*-1}U^*} = \mathbf{1}_{U^*_{1,0}}$ on U^*, and hence

$$\sum_{s^*\in D^*} \left| m_0\big(\omega \,\oplus\, (M^*)^{-1} s^*\big)\right|^2 = 1, \quad \forall\, \omega \in X^*.$$

To construct wavelet functions $\psi^{(1)}, \dots, \psi^{(m-1)}$ defined by (27), we need wavelet masks $m_1, \dots, m_{m-1}$. Following the proof of Lemma 20, one can easily provide identity matrix as $B = \{b_{\nu k}\}_{\nu,k=0}^{m-1}$ setting $m_\nu(\omega) = \mathbf{1}_{U^*_{1,\nu}}(\omega)$ for $\omega \in U^*$. By Theorem 22, the functions $\psi^{(1)}, \dots, \psi^{(m-1)}$ generate a Parseval wavelet frame. It is not difficult to check that the system $\{\psi_{jk}^{(\nu)}\}_{\nu,j,k}$ is an orthogonal basis (see also [22]). Hence, every $f \in L^2(X)$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} \sum_{\nu=1}^{m-1} \langle f, \psi_{j,k}^{(\nu)}\rangle \psi_{j,k}^{(\nu)},$$

where the convergence is unconditional.

Example 2. Let $m = n = 3$. To construct a Parseval wavelet frame generated by wavelet functions from $\mathcal{S}(X)$, we need to have a Walsh polynomial m_0 as in Theorem 22. By Theorem 12, a Walsh polynomial $m_0 \in \mathcal{M}_0^{(n)}$ is defined by its values $m_0(\mathbf{0}, \xi_1\xi_2\xi_3)$ for all vectors (ξ_1, ξ_2, ξ_3), $\xi_j \in D^*$, that correspond to the vertices of the tree in Fig. 2. As it was explained in the end of Sec. 3, we must provide at least one zero of m_0 for every branch of the tree. First, let us consider the simplest way, choosing zeros corresponding to the elements of the second column. Obviously, $\sigma_0 = \emptyset$ in this case. It follows from Lemma 18 and Theorem 10 that m_0 is the mask of a refinable function φ such that its Fourier transform $\widehat{\varphi}$ satisfies (18) with $C = 1/\sqrt{\mu(U)}$, and belongs to the space $\mathcal{S}_2^{(-2)}(X^*)$, i.e. $\widehat{\varphi}/C$ is the characteristic function of ${M^*}^{-2}U^* = U^*_{2,0}$. Hence, $m_0(U^*_{3,0}) = 1$, $m_0(U^*_{3,1}) = m_0(U^*_{3,2}) = 0$, and $\widehat{\varphi}$ does not depend on $m_0(U^*_{3,k})$ $k = 3, \ldots, 26$. To construct the wavelet functions $\psi^{(1)}, \psi^{(2)}$ defined by (27), we need wavelet masks m_1, m_2. Obviously, $\psi^{(\nu)}$, $\nu = 1, 2$, do not depend on the values $m_\nu(U^*_{3,k})$, $k = 3, \ldots, 26$, respectively. Setting $m_1(U^*_{3,1}) = m_2(U^*_{3,2}) = 1$ and $m_1(U^*_{3,2}) = m_2(U^*_{3,1}) = m_1(U^*_{3,0}) = m_2(U^*_{3,0}) = 0$, we get $\widehat{\psi}^{(1)} = \mathbf{1}_{U^*_{2,1}}$, $\widehat{\psi}^{(2)} = \mathbf{1}_{U^*_{2,2}}$. Using Theorems 22, 6, 13 and Proposition 7, it is easy to see that the wavelet system generated by $\psi^{(1)}, \psi^{(2)}$ is an orthogonal basis that after normalization may be considered as an analog of the Haar basis.

Now let us choose zeros of m_0 as it was described in the end of Sec. 3 (corresponding to the circled elements in Fig. 2), and let m_0 be non-zero in all points, corresponding to the elements of first three columns of the tree. The function m_0 is not defined yet in eight points corresponding to the elements of fourth column. Add zeros in the points corresponding to 110, 210, 212 and 220, and let m_0 be non-zero in the remaining four points. So, the function m_0 is identically zero on the 14 sets $U^*_{3,k}$, $k = 0, \ldots, 26$, and non-zero on the other ones. Setting $m_0(U^*_{3,k}) = 1$ for $k = 0, 1, 2, 3, 8$, and $m_0(U^*_{3,k}) = \sqrt{2}/2$ for $k = 4, 5, 6, 7, 14, 15, 22, 25$ we have

$$|m_0(\mathbf{0}, \mathbf{0}\xi_2\xi_3)|^2 + |m_0(\mathbf{0}, s_1^*\xi_2\xi_3)|^2 + |m_0(\mathbf{0}, s_2^*\xi_2\xi_3)|^2 = 1, \quad \forall\, \xi_2, \xi_3 \in D^*,$$

which is equivalent to

$$\sum_{s^* \in D^*} \left| m_0\big(\omega \,\oplus\, (M^*)^{-1} s^*\big) \right|^2 = 1, \quad \forall\, \omega \in X^*. \tag{38}$$

It is not difficult to check that $(s_2^*, s_2^*, s_1^*, s_1^*, s_2^*, \mathbf{0}) \in \sigma_5$, and $\sigma_6 = \emptyset$. It follows from Lemma 18 and Theorem 10 that m_0 is the mask of a refinable function φ such that its Fourier transform $\widehat{\varphi}$ satisfies (18) with $C = (\mu U)^{-1/2}$ and belongs to the space $\mathcal{S}_2^{(4)}(X^*)$. Hence, we have

$$\widehat{\varphi} = \sum_{h \in H_4^*} \sum_{s \in D^*} \sum_{s' \in D^*} \alpha_{h,s,s'} \mathbf{1}_{{M^*}^{-2}(U^*) + h + \mathbf{0}, ss'}, \tag{39}$$

where

$$\alpha_{h,s,s'} = \widehat{\varphi}(h + \mathbf{0}, ss') = (\mu U)^{-1/2} \prod_{j=1}^{5} m_0\big({M^*}^{-j}(h + \mathbf{0}, ss')\big).$$

The sum in (39) looks huge, but most of its terms are zero, there are only 19 non-zero ones.

To construct the wavelet functions $\psi^{(1)}, \psi^{(2)}$ defined by (27), we need wavelet masks m_1, m_2. Using the scheme described in the proof of Lemma 20, for every fixed digits $\xi_2, \xi_3 \in D^*$, we set $b_{0k} = m_0(\mathbf{0},\ s_k^*\xi_2\xi_3)$, then find numbers $b_{\nu k}$, $k = 0, 1, 2$, $\nu = 1, 2$, such that the matrix $B = \{b_{\nu k}\}_{\nu,k=0}^2$ is orthogonal, and set

$$m_\nu(\mathbf{0}, s_k^*\xi_2\xi_3) := b_{\nu k} = b_{\nu k}(\xi_2, \xi_3).$$

It is easy to see that the functions m_1, m_2 take only values $0, 1$ and $\pm\sqrt{2}/2$. For example, for $\xi_2 = \xi_3 = \mathbf{0}$ we have $b_{10} = b_{12} = b_{20} = b_{21} = 0$, $b_{11} = b_{22} = 1$, i.e. $m_1(U_{3,0}) = m_1(U_{3,18}) = m_2(U_{3,0}) = m_2(U_{3,9}) = 0$ and $m_1(U_{3,9}) = m_2(U_{3,18}) = 1$; for $\xi_2 = \xi_3 = s_1^*$ we have $b_{10} = b_{12} = b_{21} = 0$, $b_{11} = 1$, $b_{20} = \sqrt{2}/2$, $b_{22} = -\sqrt{2}/2$, i.e. $m_1(U_{3,4}) = m_1(U_{3,22}) = m_2(U_{3,13}) = 0$, $m_1(U_{3,13}) = 1$ and $m_2(U_{3,4}) = -m_2(U_{3,22}) = \sqrt{2}/2$. The corresponding wavelet functions are defined by (27), i.e.

$$\begin{aligned}
\widehat{\psi^{(\nu)}} &= m_\nu(M^{*-1}\cdot)\widehat{\varphi}(M^{*-1}\cdot) \\
&= \sum_{h\in H_4^*}\sum_{k=0}^{2}\sum_{s'\in D^*} \alpha_{h,s_k^*,s'} \mathbf{1}_{M^{*-2}(U^*)+h+\mathbf{0},s_k^*s'}(M^{*-1}\cdot)m_\nu(M^{*-1}\cdot) \\
&= \sum_{h\in H_4^*}\sum_{k=0}^{2}\sum_{s'\in D^*} \alpha_{h,s_k^*,s'} \sum_{\xi\in D^*} b_{\nu k}(s',\xi)\mathbf{1}_{M^{*-3}(U^*)+h+\mathbf{0},s_k^*s'\xi}(M^{*-1}\cdot) \\
&= \sum_{h\in H_4^*}\sum_{k=0}^{2}\sum_{s'\in D^*}\sum_{\xi\in D^*} \alpha_{h,s_k^*,s'} b_{\nu k}(s',\xi)\mathbf{1}_{M^{*-2}(U^*)+M^*h+s_k^*,s'\xi}.
\end{aligned}$$

Now, let us find the functions $\psi^{(\nu)}$ themselves. Setting $g = M^*h + s_k^*, s'\xi$, we have

$$\begin{aligned}
&\frac{1}{\mu(U^*)}\int_{X^*} \mathbf{1}_{M^{*-2}(U^*)+g}(\omega)\chi(x,\omega)\,d\mu(\omega) \\
&\quad = \frac{1}{\mu(U^*)}\int_{X^*} \mathbf{1}_{M^{*-2}(U^*)}(\omega)\chi(x,\omega+g)\,d\mu(\omega) \\
&\quad = \frac{1}{\mu(U^*)}\int_{M^{*-2}(U^*)} \chi(x,\omega\oplus g)\,d\mu(\omega) \\
&\quad = \frac{\chi(x,g)}{m^2\mu(U^*)}\int_{U^*} \chi(M^{-2}x,\omega)\,d\mu(\omega) \\
&\quad = \frac{\chi(x,g)}{m^2\mu(U^*)}\int_{X^*} \mathbf{1}_{U^*}(\omega)\chi(M^{-2}x,\omega)\,d\mu(\omega).
\end{aligned}$$

Taking into account that, because of Lemma 4, there holds

$$\frac{1}{\mu(U^*)}\int_{X^*} \mathbf{1}_{U^*}(\omega)\chi(y,\omega)\,d\mu(\omega) = \mathbf{1}_U(y),$$

due to Theorem 10, we obtain

$$\psi^{(\nu)}(x) = \frac{1}{m^2}\sum_{h\in H_4^*}\sum_{k=0}^{2}\sum_{s'\in D^*}\sum_{\xi\in D^*} \alpha_{h,s_k^*,s'} b_{\nu k}(s',\xi)\chi(x, M^*h + s_k^*, s'\xi)\mathbf{1}_{M^2(U)}(x).$$

By Theorem 22, the functions $\psi^{(1)}, \psi^{(2)}$ generate a Parseval wavelet frame.

Thus, we gave a complete description of all masks generating Parseval wavelet frames consisting of test functions (test-wavelets). Also, a simple explicit method for their construction was illustrated by examples. It is not difficult to see that there are enough orthogonal wavelet bases among these frames (see, e.g., [11, Theorem 4], where a similar description was given for Vilenkin groups). These frames/bases may be useful for applications to signal processing. Some examples of wavelet systems on the half-line were already investigated in this aspect. It is shown in [9] that for processing some images, a considered wavelet system on the half-line (with $M = p$) has an advantage over the standard Haar, Daubechies, and biorthogonal 9/7 wavelets. Moreover, for some fractal signals, discrete wavelet transforms associated with test-wavelet systems have advantages over the zone coding method and the discrete Haar transform corresponding to the Haar basis (see Example 1). According to [7], this can be illustrated by encoding the values of generalized Weierstrass function:

$$\mathcal{W}_{\alpha,\beta}(x) = \sum_{k=1}^{\infty} \alpha^k e^{\beta^k \pi i x}, \quad 0 < \alpha \leq 1, \quad \beta \geq 1/\alpha.$$

Numerous orthogonal bases generated by masks $m_0 \in \mathcal{M}_0^{(n)}$ (i.e. constructed as in Lemma 20), where m_0, in turn, is determined by a vector $\boldsymbol{b} = (b_0, b_1, \ldots, b_{p^n-1})$ of its values, satisfying the condition

$$|b_l|^2 + |b_{l+p^{n-1}}|^2 + \cdots + |b_{l+(p-1)p^{n-1}}|^2 = 1, \quad 0 \leq l \leq p^{n-1} - 1, \tag{40}$$

were considered. The corresponding discrete wavelet transform $O(p, n)$ was used for processing. For each signal, a vector, that gave the minimum error, was selected, and mainly it corresponded to a test-wavelet basis. The ability to select this vector expands the known methods of adapting the applied discrete transform to the processed signal. The four initial signals in [7] were the values of the function $\mathcal{W}_{\alpha,\beta}(x)$ in 243 nodes of a uniform partition of the interval $[0, 1)$ for pairs of indices $(\alpha; \beta) \in \{(0,6; 9), (0,8; 6), (0,8; 9), (0,9; 4)\}$. The root mean square coding errors show that for all four signals the methods $O(3, 1)$ and $O(3, 2)$ have advantages over both the discrete Haar transform $H(3)$ and the zone coding method $Z(3)$ (for the definition of this method, see [14, §11.3]).

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