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MATHEMATICS

Asymptotical Separation of Harmonics by Singular Spectrum Analysis

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Abstract—The paper is devoted to studying the sufficient conditions for the asymptotical separability of distinct terms in the linear combination of harmonics by singular spectrum analysis (SSA). Namely, the series $x_0, \ldots, x_{N-1}$ with $x_n = \sum_{i=1}^{r} f_{i,n}$, where $f_{i,n} = b_i \cos(\omega_i n + \gamma_i)$ and both amplitudes $|b_i|$ and frequencies $\omega_i \in (0, 1/2)$ are pairwise different, are considered. Then, as is proved in this study, under some relationship between amplitudes $|b_i|$ and the choice of standard SSA parameters, the so-called reconstructed values $\tilde{f}_{i,n}$ prove to be very close to $f_{i,n}$ for large N. Moreover, $\max_n \left(\left|\tilde{f}_{i,n} - f_{i,n}\right|\right) = O(N^{-1})$ for any i, if $N \to \infty$.

Keywords: signal processing, singular spectral analysis, linear combination of harmonics, separability of harmonics, asymptotical analysis

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

One of the most impressive experimental results in the application of singular spectrum analysis (SSA) is the extraction of harmonics from their sum. Namely, let us consider the series $x_0, \ldots, x_{N-1}$ with

$$x_n = \sum_{i=1}^{r} f_{i,n}, \quad \text{where} \quad f_{i,n} = b_i \cos(\omega_i n + \gamma_i) \tag{1}$$

with different amplitudes $|b_i|$ and frequencies $\omega_i \in (0, 1/2)$; then, at large series sizes N and the choice of standard parameters of the SSA method for this problem (the window length is taken to be $L \approx N/2$; for reconstruction of the ith term in the sum (1), two corresponding sequential principal components are used (see [1], Section 1)), the reconstructed values $\tilde{f}_{i,n}$ obtained prove to be close to $f_{i,n}$.

For example, if we take in (1) $r = 3$, $\gamma_i = 0$, $b_1 = 1$, $b_2 = 0.8$, $b_3 = 0.6$, and $\omega_1 = \sqrt{2}/4 \approx 0.35356$, $\omega_2 = \sqrt{3}/4 \approx 0.43301$, $\omega_3 = \sqrt{5}/5 \approx 0.44721$, then, using the denotation $r_i(N) = \tilde{f}_{i,N} - f_{i,N}$ and determining the maximum errors of reconstruction as $\max_{0 \le i < N} |r_i(N)|$ and the root-mean-square errors of reconstruction by the formula $\sqrt{\sum_{0 \le i < N} r_i^2(N)/N}$, we will obtain the data of Tables 1 and 2, which confirms the presented reasoning.

As already noted, all such results are experimental, i.e., formally, they are not proved. In this study, the sufficient conditions for the asymptotic separability of harmonics at $N \to \infty$ are presented (see the exact problem formulation and the result obtained in Section 4).

It is simpler to explain these conditions at $r = 2$ with $|b_1| > |b_2|$. In this case, the results of computer experiments show that for a successful reconstruction of both terms in (1), it is sufficient (except for the condition $\omega_1 \neq \omega_2$) for the harmonics to have different (and arbitrarily close) amplitudes $|b_1|$, $|b_2|$.

In the statement proved in this study (see Theorem 3 and Conclusions) it is assumed that there is some "gap" between $|b_1|$ and $|b_2|$; i.e., the ratio of $|b_2|$ to $|b_1|$ is assumed to be sufficiently small (more precisely, it is required that $|b_2/b_1| < 0.5$). At $r > 2$, on the whole, the reasoning is similar.

Table 1. Maximum errors of reconstruction for $L = N/2$

N	1st term	2nd term	3rd term
500	0.051	0.094	0.092
1000	0.031	0.077	0.071
2000	0.026	0.064	0.055
5000	0.004	0.007	0.009

Table 2. Root-mean-square errors of reconstruction for $L = N/2$

N	1st term	2nd term	3rd term
500	0.0434	0.0209	0.0196
1000	0.0025	0.0124	0.0125
2000	0.0010	0.0069	0.0070
5000	0.0001	0.0006	0.0006

Certainly, such constraints are quite burdensome; however, it is possible to prove that all the errors of reconstruction have the order $O(N^{-1})$.

Let us now proceed to the content of this work. The key statement for proving Theorem 3 is the following.

Let us use the following denotations at $n \geq 0$, $r \geq 2$, $1 \leq k < r$, and $\omega_i \in (0, 1/2)$ with $\omega_i \neq \omega_j$ at $i \neq j$:

$$f_n = f_n^{(k)} = \sum_{i=1}^{k} \beta_i \cos(2\pi\omega_i n + \gamma_i) \quad \text{and} \quad e_n = e_n^{(k)} = \sum_{i=k+1}^{r} \beta_i \cos(2\pi\omega_i n + \gamma_i),$$
$$\text{where} \quad 1 = \beta_1 > |\beta_2| > \ldots > |\beta_k| > |\beta_{k+1}| > \ldots > |\beta_r| > 0. \tag{2}$$

Let us denote the series with elements f_n and e_n by F and E, respectively; at $0 \leq n < N - 1$, their segments $f_0, \ldots, f_{N-1}$ and $e_0, \ldots, e_{N-1}$, by F_N and E_N, respectively.

Let us consider the series $\mathrm{X}_N = \mathrm{F}_N + \delta\mathrm{E}_N$ with elements $x_n = f_n + \delta e_n$, $0 \leq n < N - 1$, where δ is the perturbation parameter.

The general problem consists in separation of the signal F_N from the sum X_N at large N by using the SSA method. According to [1] (see also [4]), the procedure is as follows.

After choosing *the window length* $L < N$, the series X_N is transformed to the Hankel (*trajectory*) matrix $\mathbf{H}(\delta)$ of the size $L \times K$ with elements $\mathbf{H}(\delta)[ij] = x_{i+j-2}$, $1 \leq i \leq L$, $1 \leq j \leq K := N - L + 1$.

Then, we perform singular expansion of the matrix $\mathbf{H}(\delta)$. In [1], this operation is called *embedding*.

Let us note that rank $\mathbf{H}(\delta) = 2r$ at $\min(L; K) \geq 2r$. Since it is assumed further that $N \to \infty$ and $L/N \to \alpha \in (0, 1)$, then we can assume that the singular expansion of the matrix $\mathbf{H}(\delta)$ consists of the sum of $2r$ elementary (i.e., with a rank 1) matrices.

Next, let us denote by $\tilde{\mathbf{H}}(\delta)$ the sum of $2k$ principal (i.e., with the maximum Frobenius norms) elementary matrices and find the Henkel matrix closest (also in the Frobenius norm) to $\tilde{\mathbf{H}}(\delta)$. Applying to this matrix the operation inverse to embedding, we obtain the series $\mathrm{F}_N(\delta) = (f_0(\delta), \ldots, f_{N-1}(\delta))$, which is considered to be the approximation to the signal F_N.

According to [1], the series $\mathrm{F}_N(\delta)$ is called the *result of reconstruction of the series* F_N *from the first* $2k$ *principal components of the series* X_N, and the series with elements $r_i(\delta) = r_i^{(k)}(\delta) = f_i - f_i(\delta)$ at $0 \leq i \leq N - 1$ is called the *series of errors of this reconstruction*.

Then, the following statement holds.

Theorem 1. *If* $N \to \infty$ *and* $L/N \to \alpha \in (0, 1)$, *then*, $\max_i \left| r_i^{(k)}(\delta) \right| = O(N^{-1})$ *at*

$$|\delta| < \delta^{(k)} := 0.5 \min\left(\frac{|\beta_{k+1}|}{|\beta_k|}, \frac{\beta_k^2}{\sum_{j=k+1}^{r} \beta_j^2} \right).$$

The proof of this fact is based on the theory developed in [2], which, in turn, is based on the fundamental results obtained by T. Kato (see [3]). For the sake of conciseness, the necessary statement, in fact, contained in [2], is formulated in Theorem 2 in Section 3; all that remains in this section is the application of Theorem 2 to the concrete conditions of the considered problem.

For this application, it would seem to be necessary to prove the numerous auxiliary statements in Section 2. Thus, several elementary identities concerning summation of products of sines and cosines are presented in Subsection 2.1; Subsection 2.2 is devoted to proving Statement 1, necessary for finding the required asymptotics of positive eigenvalues; and Subsection 2.3 contains the facts related to the asymptotic behavior of uniform norms of some matrices of growing order.

With all these statements together, we obtain the proof of Theorem 1. Section 4 is devoted to proving Theorem 3, which is the aim of this study (as was already mentioned above), giving us the sufficient conditions of separation of harmonics in terms of their amplitudes (see Eq. (17)).

2. AUXILLIARY FACTS

2.1. Elementary Identities

The following identities necessary for the further reasoning can be easily proved by representation of sines and cosines via the imaginary exponents.

Lemma 1. *Let* $\omega_1, \omega_2 \in (0, 1/2)$. *Let us use the denotations* $\cos_{j1}(\gamma) = \cos(2\pi\omega_1 j + \gamma)$, $\cos_{j2}(\gamma) = \cos(2\pi\omega_2 j + \gamma)$, $\sin_{j1}(\gamma) = \sin(2\pi\omega_1 j + \gamma)$, *and* $\sin_{j2}(\gamma) = \sin(2\pi\omega_2 j + \gamma)$. *Then*,

$$\sum_{j=0}^{n-1} \cos_{j1}(\gamma_1)\cos_{j2}(\gamma_2) = \frac{\sin(\pi(\omega_1+\omega_2)n)}{2\sin(\pi(\omega_1+\omega_2))}\cos(\pi(\omega_1+\omega_2)(n-1)+\gamma_1+\gamma_2) + \frac{1}{2}\begin{cases} \dfrac{\sin(\pi(\omega_1-\omega_2)n)}{\sin(\pi(\omega_1-\omega_2))}\cos(\pi(\omega_1-\omega_2)(n-1)+\gamma_1-\gamma_2) & \text{at } \omega_1 \ne \omega_2, \\ n\cos(\gamma_1-\gamma_2) & \text{at } \omega_1 = \omega_2, \end{cases} \tag{3}$$

$$\sum_{j=0}^{n-1} \sin_{j1}(\gamma_1)\sin_{j2}(\gamma_2) = -\frac{\sin(\pi(\omega_1+\omega_2)n)}{2\sin(\pi(\omega_1+\omega_2))}\cos(\pi(\omega_1+\omega_2)(n-1)+\gamma_1+\gamma_2) + \frac{1}{2}\begin{cases} \dfrac{\sin(\pi(\omega_1-\omega_2)n)}{\sin(\pi(\omega_1-\omega_2))}\cos(\pi(\omega_1-\omega_2)(n-1)+\gamma_1-\gamma_2) & \text{at } \omega_1 \ne \omega_2, \\ n\cos(\gamma_1-\gamma_2) & \text{at } \omega_1 = \omega_2, \end{cases} \tag{4}$$

and

$$\sum_{j=0}^{n-1} \cos_{j1}(\gamma_1)\sin_{j2}(\gamma_2) = \frac{\sin(\pi(\omega_1+\omega_2)n)}{2\sin(\pi(\omega_1+\omega_2))}\sin(\pi(\omega_1+\omega_2)(n-1)+\gamma_1+\gamma_2) - \frac{1}{2}\begin{cases} \dfrac{\sin(\pi(\omega_1-\omega_2)n)}{\sin(\pi(\omega_1-\omega_2))}\sin(\pi(\omega_1-\omega_2)(n-1)+\gamma_1-\gamma_2) & \text{at } \omega_1 \ne \omega_2, \\ n\sin(\gamma_1-\gamma_2) & \text{at } \omega_1 = \omega_2. \end{cases} \tag{5}$$

2.2. Asymptotics of Eigenvalues of Matrices $\mathbf{HH}^T$ and $\mathbf{EE}^T$

Let us denote by $\mathbf{H}$ and $\mathbf{E}$ the Hankel matrices with a size $L \times K$ constructed from the series F_N and E_N in the same way as the matrix $\mathbf{H}(\delta)$ is constructed from the series X_N. If L and K are quite large, then $\operatorname{rank}\mathbf{H} = 2k$ and $\operatorname{rank}\mathbf{E} = 2(r-k)$.

Further, we will need the results on the behavior of the positive eigenvalues of the matrices $\mathbf{HH}^\mathrm{T}$ and $\mathbf{EE}^\mathrm{T}$ at large values of the series size N. These results are obtained with the use of Lemma 2.1 of [5]. Let us first formulate this lemma.

Let us consider the matrix $\mathbf{G} : \mathbb{R}^K \mapsto \mathbb{R}^L$ and use the denotation $d = \operatorname{rank}\mathbf{G}$. Let us assume that $\mathbf{G} = \sum_{k=1}^{d} P_k Q_k^\mathrm{T}$, where $P_k \in \mathbb{R}^L$ and $Q_k \in \mathbb{R}^K$, while the vectors $P_1, \ldots, P_d$ (and the vectors $Q_1, \ldots, Q_d$) are linearly independent. Let us use the denotation $X_i = P_i/\|P_i\|$, $Y_i = Q_i/\|Q_i\|$,

$$\mathbf{X} = [X_1 : \ldots : X_d], \quad \mathbf{Y} = [Y_1 : \ldots : Y_d],$$

$\mathbf{U} = [P_1 : \ldots : P_d]$, and $\mathbf{V} = [Q_1 : \ldots : Q_d]$. In addition, let us set

$$\Pi_P = \begin{pmatrix} \|P_1\| & 0 & \ldots & 0 \\ 0 & \|P_2\| & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \|P_d\| \end{pmatrix}, \quad \Pi_Q = \begin{pmatrix} \|Q_1\| & 0 & \ldots & 0 \\ 0 & \|Q_2\| & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \|Q_d\| \end{pmatrix},$$

and $\Pi_{PQ} = \Pi_P \Pi_Q$. Finally, let us use the denotation $\mathbf{C} = \mathbf{X}^\mathrm{T}\mathbf{X}\Pi_{PQ}\mathbf{Y}^\mathrm{T}\mathbf{Y}\Pi_{PQ}$.

Lemma 2. *Let λ be the positive eigenvalue of the matrix $\mathbf{GG}^\mathrm{T}$ corresponding to the eigenvector Z. Then λ is the eigenvalue of the matrix $\mathbf{C}$ corresponding to the non-zero eigenvector $\mathbf{X}^\mathrm{T}Z$.*

As was already said, the proof of this lemma can be found in ([5], Lemma 2.1). Let us apply its result to our purposes.

Let us consider the series $y_n = \sum_{\ell=1}^{p} \Delta_\ell \cos(2\pi\omega_\ell n + \gamma_\ell)$ with $\omega_\ell \in (0, 1/2)$, $\omega_\ell \neq \omega_j$ at $\ell \neq j$, and $|\Delta_1| \geq |\Delta_2| \geq \ldots \geq |\Delta_p| > 0$. Using the denotations $\cos_{k\ell}(\varphi_\ell) = \cos(2\pi k\omega_\ell + \varphi_\ell)$ and $\sin_{k\ell}(\varphi_\ell) = \sin(2\pi k\omega_\ell + \varphi_\ell)$, we set

$$C_{j\ell} = (\cos_{0\ell}(\psi_\ell), \ldots, \cos_{j-1\ell}(\psi_\ell))^\mathrm{T}, \quad S_{j\ell} = (\sin_{0\ell}(\psi_\ell), \ldots, \sin_{j-1\ell}(\psi_\ell))^\mathrm{T},$$

where $\psi_\ell = \gamma_\ell/2$. In addition, let us introduce the following denotations:

$$X_\ell = \begin{cases} \operatorname{sign}\Delta_\ell C_{L\ell}/\|C_{L\ell}\| & \text{at } 1 \leq \ell \leq p, \\ -\operatorname{sign}\Delta_{p-\ell} S_{L\ell-r}/\|S_{L\ell-r}\| & \text{at } p \leq \ell \leq 2p, \end{cases}$$

$$Y_\ell = \begin{cases} C_{K\ell}/\|C_{K\ell}\| & \text{at } 1 \leq \ell \leq p, \\ S_{K\ell-p}/\|S_{K\ell-p}\| & \text{at } p < \ell \leq 2p, \end{cases}$$

$\mathbf{X} = [X_1, \ldots, X_{2p}]$, $\mathbf{Y} = [Y_1, \ldots, Y_{2p}]$, $\mathbf{C}_{L,X} = \mathbf{X}^\mathrm{T}\mathbf{X}$ and $\mathbf{C}_{K,Y} = \mathbf{Y}^\mathrm{T}\mathbf{Y}$. Next, let us introduce $2r \times 2r$ diagonal matrices $\mathbf{\Pi}_{P,L}$, $\mathbf{\Pi}_{Q,K}$, and $\mathbf{D}$ with elements

$$[\mathbf{\Pi}_{P,L}]_{\ell,\ell} = \begin{cases} \Delta_\ell \|C_{L\ell}\| & \text{at } 1 \leq \ell \leq p, \\ -\Delta_{p-\ell} \|S_{Lp-\ell}\| & \text{at } p < \ell \leq 2p, \end{cases}$$

$$[\mathbf{\Pi}_{Q,K}]_{\ell,\ell} = \begin{cases} \|C_{K\ell}\| & \text{at } 1 \leq \ell \leq p, \\ \|S_{Kp-\ell}\| & \text{at } p < \ell \leq 2p, \end{cases}$$

$$\mathbf{D}[\ell,\ell] = \begin{cases} \Delta_\ell & \text{at } 1 \leq \ell \leq p, \\ -\Delta_{p-\ell} & \text{at } p < \ell \leq 2p, \end{cases}$$

and set $\mathbf{\Pi}_{PQ} = \mathbf{\Pi}_{P,L}\mathbf{\Pi}_{Q,K}$. Finally, let $\mathbf{G}$ be a trajectory matrix of the series y_n with a size $L \times K$; $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2p}$ are positive eigenvalues of the matrix $\mathbf{GG}^\mathrm{T}$.

Statement 1. *If* $\min(L, K) \to \infty$ *at* $N \to \infty$, *then* $\lambda_i/LK \to \Delta^2_{\lceil i/2 \rceil}/4$.

Proof. Since

$$y_{n+m} = \sum_{\ell=1}^{p} \Delta_\ell \cos(2\pi n\omega_\ell + \psi_\ell)\cos(2\pi m\omega_\ell + \psi_\ell) - \sum_{\ell=1}^{p} \Delta_\ell \sin(2\pi n\omega_\ell + \psi_\ell)\sin(2\pi m\omega_\ell + \psi_\ell),$$

$\mathbf{G}$ can be represented in the form

$$\mathbf{G} = \sum_{\ell=1}^{p} \Delta_\ell C_{L\ell} C_{K\ell}^{\mathrm{T}} - \sum_{\ell=1}^{p} \Delta_\ell S_{L\ell} S_{K\ell}^{\mathrm{T}} = \mathbf{X}\mathbf{\Pi}_{PQ}\mathbf{Y}^{\mathrm{T}}.$$

According to Lemma 2, the set of positive eigenvalues of the matrix $\mathbf{G}\mathbf{G}^{\mathrm{T}}$ coincides with the spectrum of the matrix $\mathbf{C} = \mathbf{C}_{L,X}\mathbf{\Pi}_{PQ}\mathbf{C}_{K,Y}\mathbf{\Pi}_{PQ}$.

Since the elements of the matrix $\mathbf{C}_{K,Y}$ have the form

$$c_{\ell m}^{(K,Y)} = \begin{cases} C_{K\ell}^{\mathrm{T}} C_{Km} / \|C_{K\ell}\| \|C_{Km}\| & \text{at } \ell, m \le p, \\ S_{K\ell}^{\mathrm{T}} S_{Km} / \|S_{K\ell}\| \|S_{Km}\| & \text{at } \ell, m > p, \\ C_{K\ell}^{\mathrm{T}} S_{Km} / \|C_{K\ell}\| \|S_{Km}\| & \text{at } \ell \le p \text{ and } m > p, \\ C_{Km}^{\mathrm{T}} S_{K\ell} / \|C_{Km}\| \|S_{K\ell}\| & \text{at } \ell > p \text{ and } m \le p, \end{cases}$$

then, according to Lemma 1, the matrices $\mathbf{Y}^{\mathrm{T}}\mathbf{Y} = \mathbf{C}_{K,\,Y}$ converge elementwise at $K \to \infty$ to the unit matrix $\mathbf{I}_{2p}$ with a size $2p \times 2p$. Similarly, the matrices $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{C}_{L,\,X}$ converge to $\mathbf{I}_{2p}$ at $L \to \infty$.

Next, $\sqrt{2}\Pi_{P,L}/\sqrt{L} \to \mathbf{D}$ as $L \to \infty$ and, accordingly, $\sqrt{2}\Pi_{Q,K}/\sqrt{K} \to \mathbf{I}_{2r}$ as $K \to \infty$. This mean that $4\mathbf{C}/KL \to \mathbf{D}^2$ as L , $K \to \infty$.

Since $\mathbf{D}^2$ is a diagonal matrix with diagonal elements

$$\Delta_1^2, \ldots, \Delta_p^2, \Delta_1^2, \ldots, \Delta_p^2,$$

and $\Delta_1^2 \ge \ldots \ge \Delta_p^2 > 0$, then Statement 1 is proved.

□

Corollary 1. *Let us consider the trajectory matrices* $\mathbf{H}$ *and* $\mathbf{E}$ *with a size* $L \times K$ *constructed from the series* F_N *and* E_N, *respectively. Let* $L \sim \alpha N$ *with* $\alpha \in (0, 1)$ *at* $N \to \infty$.

Let us denote by $\mu_1, \ldots, \mu_{2k}$ *and* $\mu_{2k+1}, \ldots, \mu_{2r}$ *the positive eigenvalues of the matrices* $\mathbf{H}\mathbf{H}^{\mathrm{T}}$ *and* $\mathbf{E}\mathbf{E}^{\mathrm{T}}$, *respectively. Then, according to Statement* 1, *at* $1 \le i \le 2r$,

$$\mu_i/N^2 \to \alpha(1-\alpha)\beta^2_{\lceil i/2 \rceil}/4.$$

In particular, if we denote by $\mu_{\max}$ *and* $\mu_{\min}$ *the maximum and minimum eigenvalues of the matrix* $\mathbf{H}\mathbf{H}^{\mathrm{T}}$, *respectively, then it will appear that*

$$\mu_{\max} \sim N^2\alpha(1-\alpha)/4 \quad \textit{and} \quad \mu_{\min} \sim \beta_k^2 N^2 \alpha(1-\alpha)/4.$$

Meanwhile, all μ_i *at* $1 \le i \le 2r$ *have the order of growth* N^2, *while* $\|\mathbf{H}\|$ *and* $\|\mathbf{E}\|$ *grow linearly with respect to* N *at* $N \to \infty$.

2.3. On the Asymptotics of the Uniform Norm of Some Matrices

In this study, we use two matrix norms. *The spectral norm* $\|\mathbf{K}\|$ of the matrix $\mathbf{K}$ is its maximum singular number; in other words, $\|\mathbf{K}\|^2$ is the maximum eigenvalue of the matrix $\mathbf{K}\mathbf{K}^{\mathrm{T}}$.

The uniform norm $\|\mathbf{K}\|_{\max}$ is the maximum of the moduli of the elements $L \times K$ of the matrix $\mathbf{K}$. These norms are linked by the following inequalities (see [7], Subsection 2.3.2).

$$\|\mathbf{K}\|_{\max} \le \|\mathbf{K}\| \le \sqrt{LK}\,\|\mathbf{K}\|_{\max}. \tag{6}$$

In addition, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are the matrices with sizes $L \times K$ and $K \times M$, respectively, then

$$\|\mathbf{K}_1\mathbf{K}_2\|_{\max} \le K\|\mathbf{K}_1\|_{\max}\|\mathbf{K}_2\|_{\max}. \tag{7}$$

In this subsection, the estimates of the rate of convergence of the uniform norms of some trajectory matrices for large series are presented.

Since $\mathbf{H} = \sum_{i=1}^{k} \beta_i \mathbf{H}_i$, where $\mathbf{H}_i$ is the Hankel matrix generated by the series $f_{in} = \cos(2\pi\omega_i n + \gamma_i)$ with $i \le k$, then $\|\mathbf{H}\|_{\max} \le \sum_{i=1}^{k} |\beta_i|$. Similarly, $\|\mathbf{E}\|_{\max} \le \sum_{i=k+1}^{r} |\beta_i|$.

Lemma 3. *Let us consider the singular expansion of the matrix* $\mathbf{H}$:

$$\mathbf{H} = \sum_{i=1}^{2k} \sqrt{\mu_i} U_i V_i^{\mathrm{T}},$$

while we assume that $L/N \to \alpha \in (0, 1)$. *Let us denote by* $\mathbf{P}_{\mu_i} = U_i U_i^{\mathrm{T}}$ *the orthogonal projector onto the one-dimensional space generated by the eigenvector* U_i *of the matrix* $\mathbf{H}\mathbf{H}^{\mathrm{T}}$ *with the eigenvalue* μ_i, $i \le 2k$.

Then, $\|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = O(N^{-1})$, $\|\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(1)$, *and* $\left\|\mathbf{P}_{\mu_i}\mathbf{H}\mathbf{E}^{\mathrm{T}}\right\|_{\max} = O(1)$.

Proof. Let us start from $\|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}$. Let us use at $1 \le j \le k$ the following denotations:

$$P_{j1} = (1, \cos(2\pi\omega_j), \ldots, \cos(2\pi\omega_j(L-1)))^{\mathrm{T}},$$

$$P_{j2} = (0, \sin(2\pi\omega_j), \ldots, \sin(2\pi\omega_j(L-1)))^{\mathrm{T}},$$

$$Q_{j1} = (1, \cos(2\pi\omega_j), \ldots, \cos(2\pi\omega_j(K-1)))^{\mathrm{T}},$$

$$Q_{j2} = (0, \sin(2\pi\omega_j), \ldots, \sin(2\pi\omega_j(K-1)))^{\mathrm{T}}.$$

Then $U_i = \sum_{j=1}^{k} a_{ij1} P_{j1} / \|P_{j1}\| + \sum_{j=1}^{k} a_{ij2} P_{j2} / \|P_{j2}\|$, while,

$$\begin{aligned} 1 - \|U_i\|^2 &= \sum_{j=1}^{k} a_{ij1}^2 + \sum_{j=1}^{k} a_{ij2}^2 + 2\sum_{m \ne \ell} a_{im1} a_{i\ell 1} (P_{m1}, P_{\ell 1}) / \|P_{m1}\| \|P_{\ell 1}\| \\ &+ 2\sum_{m \ne \ell} a_{im2} a_{i\ell 2} (P_{m2}, P_{\ell 2}) / \|P_{m2}\| \|P_{\ell 2}\| + 2\sum_{1 \le m, \ell \le k} a_{im1} a_{i\ell 2} (P_{m1}, P_{\ell 2}) / \|P_{m1}\| \|P_{\ell 2}\|. \end{aligned} \quad (8)$$

Since, according to Lemma 1, all the absolute values of the scalar products in (8) are bounded and the norms $\|P_{mi}\|$ grow as $\sqrt{N}$, then the coefficients a_{ij1} and a_{ij2} are also bounded.

Next,

$$\begin{aligned} \mathbf{P}_{\mu_i} = U_i U_i^{\mathrm{T}} &= \sum_{j,m=1}^{k} a_{ij1} a_{im1} P_{j1} P_{j1}^{\mathrm{T}} / \|P_{j1}\| \|P_{m1}\| + \sum_{j,m=1}^{k} a_{ij2} a_{im2} P_{j2} P_{m2}^{\mathrm{T}} / \|P_{j2}\| \|P_{m2}\| \\ &+ 2\sum_{j,m=1}^{k} a_{ij1} a_{im2} \left(P_{j1} P_{j2}^{\mathrm{T}} + P_{m2} P_{j1}^{\mathrm{T}}\right) / \|P_{j1}\| \|P_{m2}\|. \end{aligned}$$

It should be noted that $\mathbf{E} = \sum_{i=k+1}^{r} \beta_i \mathbf{E}_i$. Since according to inequality (7), $\left\|P_{j\ell} P_{mq}^{\mathrm{T}} \mathbf{E}_i\right\|_{\max} \le \|P_{j\ell}\|_{\max} \left\|P_k^{\mathrm{T}} \mathbf{E}_i\right\|_{\max}$ and $\left\|P_{mq}^{\mathrm{T}} \mathbf{E}\right\|_{\max} = O(1)$ by virtue of Lemma 1, then, the first statement is proved.

Next, $V_i = \sum_{j=1}^{k} b_{ij1} Q_{j1} / \|Q_{j1}\| + \sum_{j=1}^{k} b_{ij2} Q_{j2} / \|Q_{j2}\|$, where, as in the first case, $|b_{ij1}| + |b_{ij2}| = O(1)$ at $N \to \infty$ for any i, j. Since

$$\begin{aligned} \mathbf{P}_{\mu_i}\mathbf{H} = \sqrt{\mu_i} U_i V_i^{\mathrm{T}} &= \sqrt{\mu_i} \left(\sum_{j=1}^{k} \sum_{m=1,2} a_{ijm} P_{jm} / \|P_{jm}\| \right) \left(\sum_{\ell=1}^{k} \sum_{p=1,2} b_{i\ell p} Q_{\ell p} / \|Q_{\ell p}\| \right)^{\mathrm{T}} \\ &= \sqrt{\mu_i} \sum_{j,\ell=1}^{k} \sum_{m,p=1,2} a_{ijm} b_{i\ell p} P_{jm} Q_{\ell p}^{\mathrm{T}} / \|P_{jm}\| \|Q_{\ell p}\|, \end{aligned}$$

and $\left\|P_{jm} Q_{\ell p}^{\mathrm{T}}\right\|_{\max} = O(1)$ and $\sqrt{\mu_i} / \|P_{jm}\| \|Q_{\ell p}\| = O(1)$ at $N \to \infty$, then $\|\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(1)$.

Similarly, since

$$\mathbf{P}_{\mu_i}\mathbf{H}\mathbf{E}^{\mathrm{T}} = \sqrt{\mu_i}\sum_{q=k+1}^{r}\beta_q\sum_{j,\ell=1}^{k}\sum_{m,p=1,2} a_{ijm}b_{i\ell p}P_{jm}Q_{\ell p}^{\mathrm{T}}\mathbf{E}_q^{\mathrm{T}}/\|P_{jm}\|\|Q_{\ell p}\|,$$

$\left\|Q_m^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\right\|_{\max} = O(1)$ (see Lemma 1) and, according to inequality (7),

$$\left\|P_{jm}Q_{\ell p}^{\mathrm{T}}\mathbf{E}_q^{\mathrm{T}}\right\|_{\max} \le \|P_{jm}\|_{\max}\left\|Q_{\ell p}^{\mathrm{T}}\mathbf{E}_q^{\mathrm{T}}\right\|_{\max} = O(1),$$

then all the statements of Lemma 3 are proved.

□

Let us introduce the following denotations. $\mathrm{U}_0^{\perp}$ is the linear space of the columns of the matrix $\mathbf{H}$; $\mathbf{P}_0^{\perp}$ is the orthogonal projector onto this subspace; and $\mathbf{P}_0 = \mathbf{I}_{2k} - \mathbf{P}_0^{\perp}$ is the orthogonal projector onto the orthogonal complement U_0 to $\mathrm{U}_0^{\perp}$.

Lemma 4. *Let $L/N \to \alpha \in (0, 1)$ at $N \to \infty$. Then,*

1. $\|\mathbf{H}\mathbf{E}^{\mathrm{T}}\|_{\max} = \|\mathbf{E}\mathbf{H}^{\mathrm{T}}\|_{\max} = O(1)$.

2. *There are the constants C_f and C_e such that*

$$\left\|\mathbf{H}\mathbf{H}^{\mathrm{T}}\right\|_{\max} \le 0.5K\sum_{i=1}^{k}\beta_i^2 + C_f = O(N) \tag{9}$$

and

$$\left\|\mathbf{E}\mathbf{E}^{\mathrm{T}}\right\|_{\max} \le 0.5K\sum_{i=k+1}^{k}\beta_i^2 + C_e = O(N). \tag{10}$$

3. $\|\mathbf{B}(\delta)\|_{\max} = O(N)$.

4. $\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N)$.

5. $\|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} = O(N^{-1})$.

6. $\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N)$.

7. $\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = O(N)$.

8. $\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = O(N)$.

Proof. 1. This statement follows from the relationship $\mathbf{H}\mathbf{E}^{\mathrm{T}} = \sum_{i=1}^{k}\sum_{j=k+1}^{r}\beta_i\beta_j\mathbf{H}_i\mathbf{E}_j^{\mathrm{T}}$; according to equality (3) for the case $\omega_1 \ne \omega_2$, $\left\|\mathbf{H}_i\mathbf{E}_j^{\mathrm{T}}\right\|_{\max} = O(1)$.

2. This statement follows from the equality $\mathbf{H}\mathbf{H}^{\mathrm{T}} = \sum_{i=1}^{k}\sum_{j=1}^{k}\beta_i\beta_j\mathbf{H}_i\mathbf{H}_j^{\mathrm{T}}$ and from

$$\left\|\mathbf{H}_i\mathbf{H}_i^{\mathrm{T}}\right\|_{\max} \le 0.5K + 0.5\frac{1}{\sin(\pi\omega_i)},$$

and at $i \ne j$,

$$\left\|\mathbf{H}_i\mathbf{H}_j^{\mathrm{T}}\right\|_{\max} \le 0.5\left(\frac{1}{\sin\left(\pi|\omega_i - \omega_j|\right) + \sin\left(\pi|\omega_i + \omega_j|\right)}\right)$$

(see Eq. (3) of Lemma 1 at $\omega_1 = \omega_2$ and $\omega_1 \ne \omega_2$). For the case $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ the reasoning is similar.

3. This statement follows from the definition of $\mathbf{B}(\delta)$ and the already proved relationships.

4. Using inequality (7) and Lemma 3, we obtain

$$\begin{aligned}\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} &\le |\delta|\left\|\mathbf{H}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\right\|_{\max} + |\delta|\left\|\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\right\|_{\max} + \delta^2\left\|\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\right\|_{\max} \\ &\le 2|\delta|L\left\|\mathbf{H}\mathbf{E}^{\mathrm{T}}\right\|_{\max}\|\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} + \delta^2L^2\|\mathbf{E}\|_{\max}\|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}\|\mathbf{H}\|_{\max} = O(N).\end{aligned}$$

5. Since $\mathbf{P}_0^{\perp} = \sum_{i=1}^{2k} \mathbf{P}_{\mu_i}$, then the required relationship follows from Lemma 3.

6. Note that $\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} \le \|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} + \|\mathbf{P}_0^{\perp}\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max}$. Next,

$$\|\mathbf{P}_0^{\perp}\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} \le |\delta| \, \|\mathbf{H}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max}$$
$$+ |\delta| \, \|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} + \delta^2 \|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max}.$$

Since

$$\|\mathbf{H}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} \le L \|\mathbf{H}\mathbf{E}^{\mathrm{T}}\|_{\max} \|\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N),$$
$$\|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{H}\|_{\max} \ge L^2 \|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} \|\mathbf{H}^{\mathrm{T}}\|_{\max} \|\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N),$$
$$\|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} \le L^2 \|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} \|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} \|\mathbf{H}\|_{\max} = O(1),$$

then $\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H}\|_{\max} = O(N)$ and, thus, the statement is proved.

7. This statement follows from the inequality $\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} \le L \|\mathbf{B}(\delta)\|_{\max} \|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}$ and Lemma 3.

8. Since $\mathbf{P}_0 = \mathbf{I}_{2k} - \mathbf{P}_0^{\perp}$, then

$$\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} \le \|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} + \|\mathbf{P}_0^{\perp}\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}.$$

The first term is already studied (see item 4 of this lemma). As for the second term, we get

$$\|\mathbf{P}_0^{\perp}\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} \le |\delta| \, \|\mathbf{P}_0^{\perp}\mathbf{H}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} + |\delta| \, \|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}$$
$$+ \delta^2 \|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = |\delta| J_1 + |\delta| J_2 + \delta^2 J_3.$$

Next,

$$J_1 = \|\mathbf{H}\mathbf{E}^{\mathrm{T}}\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} \le L \|\mathbf{H}\mathbf{E}^{T}\|_{\max} \|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = O(1),$$
$$J_2 \le L^2 \|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} \|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} \|\mathbf{H}\|_{\max} = O(1)$$

and, similarly, $J_3 = O(1)$, which implies the required.

□

3. THE MAIN THEOREM AND ITS APPLICATION

Following [2], let us use the denotations $\mathbf{A}^{(1)} = \mathbf{H}\mathbf{E}^{\mathrm{T}} + \mathbf{E}\mathbf{H}^{\mathrm{T}}$, $\mathbf{A}^{(2)} = \mathbf{E}\mathbf{E}^{\mathrm{T}}$, $\mathbf{B}(\delta) = \delta\mathbf{A}^{(1)} + \delta^2\mathbf{A}^{(2)}$. Let $\mu_{\min}$ be the minimum positive eigenvalue of the matrix $\mathbf{H}\mathbf{H}^{\mathrm{T}}$, and $B(\delta) := |\delta| \|\mathbf{A}^{(1)}\| + \delta^2 \|\mathbf{A}^{(2)}\|$.

The following statement is the principal to obtain the final result, i.e., Theorem 3. Let us use the denotation $\mathbf{A}_0^{(2)} = \mathbf{P}_0\mathbf{A}^{(2)}\mathbf{P}_0$ and let $\mathbf{S}_0$ be the Moor–Penrose pseudoinverse matrix to the matrix $\mathbf{H}\mathbf{H}^{\mathrm{T}}$ with $\|\mathbf{S}_0\| = 1/\mu_{\min}$. As in Lemma 3, let $\mathbf{P}_{\mu_i}$ be the orthogonal projector onto the one-dimensional space generated by the eigenvector U_i of the matrix $\mathbf{H}\mathbf{H}^{\mathrm{T}}$ with the eigenvalue μ_i, while $\mu_1 = \mu_{\max}$ and $\mu_{2k} = \mu_{\min}$.

Theorem 2. *Let us assume that there is $\delta_k > 0$ such that $B(\delta_k) \le \mu_{\min}/4$. Then, at $|\delta| < \delta_k$, the inequality $\|\delta\mathbf{A}_0^{(2)}/\mu_{\min}\| < 1$ is valid, and the matrix $\mathbf{I} - \delta\mathbf{A}_0^{(2)}/\mu_{\min}$ is invertible.*

We denote as $r_i(\delta) = r_i(\delta, N)$, $0 \le i \le N-1$ the errors of reconstruction of the series F_N from the first $2k$ principal components by the SSA method.

In addition, let us set $\mathbf{L}(\delta) = \mathbf{L}_1(\delta) + \mathbf{L}_1^{\mathrm{T}}(\delta)$ with

$$\mathbf{L}_1(\delta) = \sum_{i=1}^{2k} \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0}{\mu_i} \left(\mathbf{I} - \delta\mathbf{A}_0^{(2)}/\mu_i\right)^{-1}. \quad (11)$$

Then, *at* $|\delta| < \delta_k$,

$$\max_{0\le i<N} |r_i(\delta)| \le C \frac{\|\mathbf{S}_0\mathbf{B}(\delta)\| \|\mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\|}{1-4\|\mathbf{B}(\delta)\|/\mu_{\min}} \|\mathbf{H}\| + \|\mathbf{L}(\delta)\mathbf{H}\|_{\max} + |\delta| \|\mathbf{L}(\delta)\mathbf{E}\|_{\max} + |\delta| \|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max}, \tag{12}$$

where $C > 0$ *is some absolute constant*.

Proof. The proof of this fact follows directly from ([2], Theorem 2.5 and Subsection 5.3) and from the left-hand inequality among inequalities (6). Note that inequality (12) was in fact used in ([2], Subsection 5.3.1) and in Section 3 of [6].

□

Lemma 5. *Let* $N \to \infty$ *and* $L \sim \alpha N$ *with* $\alpha \in (0, 1)$. *Then there are* $\delta_k > 0$ *and* N_k *such that at* $N > N_k$ *and* $|\delta| < \delta_k$, *the inequality* $B(\delta) < \mu_{\min}/4$ *is fulfilled*. *Meanwhile*, $1 - 4\|\mathbf{B}(\delta)\|/\mu_{\min} > 0$ *at* $|\delta| < \delta_k$ *and a sufficiently large* N.

Proof. The first statement follows from the fact the $\mu_{\min}$ and $B(\delta)$ have the same order of growth at $N \to \infty$.

Indeed, $\mu_{\min} \sim \beta_k^2\alpha(1-\alpha)N^2/4$ at $N \to \infty$. Statement 1 of Lemma 4 implies that $\|\mathbf{A}^{(1)}\|_{\max} = O(1)$, therefore, $\|\mathbf{A}^{(1)}\| = O(N)$ (see inequality (6)).

Since $\|\mathbf{A}^{(2)}\| = \mu_{2k+2} \sim \beta_{k+1}^2\alpha(1-\alpha)N^2/4$, then $B(\delta)/\mu_{\min} \to \delta^2\beta_{k+1}^2/\beta_k^2$ at any δ, which implies the required with $\delta_k = 0.5|\beta_k|/|\beta_{k+1}|$.

The second statement follows from the fact that $\|\mathbf{B}(\delta)\| \le B(\delta)$.

□

Next, it is necessary to obtain the upper estimates for all the terms in the right-hand part of (12). For that, we will need to prove some lemmas; in all of them, we will assume that $N \to \infty$ and $L/N \to \alpha \in (0, 1)$.

Lemma 6. *If* $N \to \infty$ *and* $L/N \to \alpha \in (0, 1)$, *then* $\|\mathbf{S}_0\mathbf{B}(\delta)\| = O(N^{-1})$.

Proof. Since $\|\mathbf{S}_0\| = 1/\mu_{\min} \asymp N^{-2}$ and, as was already noted, $\|\mathbf{A}^{(1)}\| = O(N)$, we need to determine the norm of $\|\mathbf{S}_0\mathbf{A}^{(2)}\|$.

Since $\mathbf{H}\mathbf{H}^{\mathrm{T}} = \sum_{i=1}^{2k}\mu_i\mathbf{P}_{\mu_i}$, then $\mathbf{S}_0 = \sum_{i=1}^{2k}\mathbf{P}_{\mu_i}/\mu_i$ and

$$\|\mathbf{S}_0\mathbf{E}\mathbf{E}^{\mathrm{T}}\| \le \sum_{i=1}^{2k}\|\mathbf{P}_{\mu_i}\mathbf{E}\mathbf{E}^T\|/\mu_i \le \sum_{i=1}^{2k}\|\mathbf{P}_{\mu_i}\mathbf{E}\|\|\mathbf{E}\|/\mu_i \le \sum_{i=1}^{2k}\frac{\sqrt{LK}}{\mu_i}\|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}\|\mathbf{E}\| = O(N^{-1}),$$

which completes the proof.

□

Corollary 2. *If* $|\delta| < \delta_k = 0.5|\beta_k|/|\beta_{k+1}|$, *then under the conditions of Lemma* 6

$$\frac{\|\mathbf{S}_0\mathbf{B}(\delta)\|\|\mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\|}{1-4\|\mathbf{B}(\delta)\|/\mu_{\min}}\|\mathbf{H}\| = O(N^{-1}).$$

Proof. According to Lemma 6,

$$\|\mathbf{S}_0\mathbf{B}(\delta)\|\|\mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\| \le \|\mathbf{S}_0\mathbf{B}(\delta)\|^2 = O(N^{-2}).$$

Since $\|\mathbf{H}\| = O(N)$ and $1 - 4\|\mathbf{B}(\delta)\|/\mu_{\min} > 0$ at $|\delta| < \delta_k$, the statement is proved.

□

Lemma 7. *Let us use the denotation* $\mathbf{Z}_i = \delta \mathbf{A}_0^{(2)}/\mu_i = \delta \mathbf{P}_0 \mathbf{E}\mathbf{E}^{\mathrm{T}} \mathbf{P}_0/\mu_i$ *at* $i = 1, \ldots, 2k$. *Then*

$$\left\| \sum_{n\geq 1} \mathbf{Z}_i^n \right\|_{\max} = O(N^{-1}) \tag{13}$$

at sufficiently small δ.

Proof. Let us prove first that $\|\mathbf{Z}_i\|_{\max} \leq |\delta| c/N$ at some $c > 0$.

Note that $\mathbf{P}_0\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_0 = \mathbf{E}\mathbf{E}^{\mathrm{T}} - \mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}} - \mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_0^{\perp} + \mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_0^{\perp}$. Next, according to item 2 of Lemma 4,

$$\left\|\mathbf{E}\mathbf{E}^{\mathrm{T}}\right\|_{\max} \leq 0.5K \sum_{j=k+1}^{r} \beta_j^2 + C_e.$$

In addition, from item 5 of Lemma 4 and inequality (7) we get

$$\left\|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}}\right\|_{\max} \leq L \left\|\mathbf{P}_0^{\perp}\mathbf{E}\right\|_{\max} \|\mathbf{E}\|_{\max} = O(1)$$

and $\left\|\mathbf{P}_0^{\perp}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_0^{\perp}\right\|_{\max} \leq L \left\|\mathbf{P}_0^{\perp}\mathbf{E}\right\|_{\max}^2 = O(N^{-1})$.

Then, since (see Corollary 1) $\mu_i/N^2 \to \alpha(1-\alpha)\beta_{\lceil i/2 \rceil}^2/4$, then

$$\|\mathbf{Z}_i\|_{\max} \leq |\delta| \left(\frac{0.5K \sum_{j=k+1}^{r} \beta_j^2}{\mu_i} + O(N^{-1}) \right) \leq |\delta| \frac{æ \sum_{j=k+1}^{r} \beta_j^2}{\alpha \beta_{\lceil i/2 \rceil}^2}$$

$$\leq |\delta| \frac{æ \sum_{j=k+1}^{r} \beta_j^2/\beta_k^2}{\alpha} \frac{1}{N}$$

at some æ > 2. Therefore, the inequality $\|\mathbf{Z}_i\|_{\max} \leq |\delta| c/N$ is proved at

$$c = \frac{æ \sum_{j=k+1}^{r} \beta_j^2/\beta_k^2}{\alpha}.$$

Next, if we take

$$|\delta| < \delta^{(k,2)} := \frac{1}{2\sum_{j=k+1}^{r} \beta_j^2/\beta_k^2} < 1/c, \tag{14}$$

we obtain the inequality $|\delta| c < 1$. Under this condition, it is easy to show that $\left\|\mathbf{Z}_i^n\right\|_{\max} \leq |\delta|^n c^n/N$ at any $n \geq 1$. Indeed, since $\left\|\mathbf{Z}_i^n\right\|_{\max} \leq L \left\|\mathbf{Z}^{n-1}\right\|_{\max} \|\mathbf{Z}\|_{\max}$ and $L\|\mathbf{Z}_i\| \leq |\delta| c$, this can be proved with the help of a simple induction. This immediately gives us the desired inequality

$$\left\| \sum_{\ell\geq 1} \mathbf{Z}_i^{\ell} \right\|_{\max} \leq \frac{|\delta| c}{1-|\delta| c} \frac{1}{N}, \tag{15}$$

and the statement is proved.

□

Now, let us consider matrices $\mathbf{L}(\delta)\mathbf{H}$.

Lemma 8. $\|\mathbf{L}(\delta)\mathbf{H}\|_{\max} = O(N^{-1})$ *at* $N \to \infty$ *and quite small* δ.

Proof. Since $\mathbf{P}_0\mathbf{H} = 0$, then $\mathbf{Z}_i\mathbf{H} = 0$ and $\mathbf{L}_1(\delta)\mathbf{H} = \mathbf{0}$. Therefore,

$$\mathbf{L}(\delta)\mathbf{H} = \mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{H} = \sum_{i=1}^{2k}\mathbf{L}_{1i}^{\mathrm{T}}(\delta)\mathbf{H} \quad \text{with} \quad \mathbf{L}_{1i}^{\mathrm{T}}(\delta) = (\mathbf{I} - \mathbf{Z}_i)^{-1}\frac{\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}}{\mu_i}, \quad \text{while}$$

$$\mathbf{L}_{1i}^{\mathrm{T}}(\delta)\mathbf{H} = \frac{\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}}{\mu_i} + \left(\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right)\frac{\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}}{\mu_i}.$$

Next, according to Statements 4 and 6 of Lemma 4, $\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N)$ and $\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N)$. Therefore, according to Lemma 7,

$$\left\|\left(\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right)\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\right\|_{\max} \le L\left\|\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right\|_{\max}\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{H}\|_{\max} = O(N),$$

and the statement is proved.

□

Lemma 9. *At a sufficiently small* δ *and* $N \to \infty$, $\|\mathbf{L}(\delta)\mathbf{E}\|_{\max} = O(N^{-1})$.

Proof. By definition, $\mathbf{L}(\delta)\mathbf{E} = \mathbf{L}_1(\delta)\mathbf{E} + \mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{E}$ with

$$\mathbf{L}_1(\delta)\mathbf{E} = \sum_{i=1}^{2k}\mathbf{L}_{1i}(\delta)\mathbf{E} = \sum_{i=1}^{2k}\frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0}{\mu_i}(\mathbf{I} - \mathbf{Z}_i)^{-1}\mathbf{E}.$$

Let us start from $\|\mathbf{L}_1(\delta)\mathbf{E}\|_{\max}$. Since

$$\mathbf{L}_{1i}(\delta)\mathbf{E} = \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0}{\mu_i}(\mathbf{I} - \mathbf{Z}_i)^{-1}\mathbf{E} = \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0}{\mu_i}\mathbf{E} + \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0}{\mu_i}\left(\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right)\mathbf{E}$$

$$= \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{E}}{\mu_i} - \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0^{\perp}\mathbf{E}}{\mu_i} + \frac{\mathbf{P}_{\mu_i}\mathbf{B}(\delta)}{\mu_i}\left(\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right)\mathbf{E}, \quad \text{and}$$

$$\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\|_{\max} \le |\delta|\left\|\mathbf{P}_{\mu_i}\mathbf{H}\mathbf{E}^{\mathrm{T}}\right\|_{\max} + |\delta|\left\|\mathbf{P}_{\mu_i}\mathbf{E}\mathbf{H}^{\mathrm{T}}\right\|_{\max} + \delta^2\left\|\mathbf{P}_{\mu_i}\mathbf{E}\mathbf{E}^{\mathrm{T}}\right\|_{\max}$$

$$\le |\delta|\left\|\mathbf{P}_{\mu_i}\mathbf{H}\mathbf{E}^{\mathrm{T}}\right\|_{\max} + |\delta| L\|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}\|\mathbf{H}\|_{\max} + \delta^2 L\|\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max}\|\mathbf{E}\|_{\max} = O(1),$$

$$\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{E}\|_{\max} \le L\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\|_{\max}\|\mathbf{E}\|_{\max} = O(N), \quad \text{and}$$

$$\left\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0^{\perp}\mathbf{E}\right\|_{\max} \le L\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\|_{\max}\left\|\mathbf{P}_0^{\perp}\mathbf{E}\right\|_{\max} = O(1),$$

then

$$\|\mathbf{L}_{1i}(\delta)\mathbf{E}\|_{\max} \le \|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{E}\|_{\max}/\mu_i + \left\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\mathbf{P}_0^{\perp}\mathbf{E}\right\|_{\max}/\mu_i$$
$$+ L^2\|\mathbf{P}_{\mu_i}\mathbf{B}(\delta)\|_{\max}\left\|\left(\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right)\right\|_{\max}\|\mathbf{E}\|_{\max}/\mu_i = O(N^{-1}).$$

Let us consider $\left\|\mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{E}\right\|_{\max}$.

$$\mathbf{L}_{1i}^{\mathrm{T}}(\delta)\mathbf{E} = (\mathbf{I} - \mathbf{Z}_i)^{-1}\frac{\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}}{\mu_i} = \frac{\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}}{\mu_i} + \left(\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right)\frac{\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}}{\mu_i}.$$

Since (see Statements 7 and 8 of Lemma 4) $\|\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = O(N)$ and $\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{P}_{\mu_i}\mathbf{E}\|_{\max} = O(N)$, and $\left\|\sum_{\ell\ge 1}\mathbf{Z}_i^{\ell}\right\| = O(N^{-1})$ (see Lemma 7), then

$$\left\|\mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{E}\right\|_{\max} \le \left\|\mathbf{L}_{11}^{\mathrm{T}}(\delta)\mathbf{E}\right\|_{\max} + \left\|\mathbf{L}_{12}^{\mathrm{T}}(\delta)\mathbf{E}\right\|_{\max} = O(N^{-1}).$$

The statement is proved.

□

With all the results of Corollary 2, Lemmas 8 and 9, as well as Statement 5 of Lemma 4, we get from inequality (12) of Theorem 2 the final result of this study, i.e., Theorem 1, presented in Introduction.

4. EXTRACTION OF HARMONIC TERMS FROM THEIR SUM

Let us proceed to the extraction of the harmonic terms from their sum. Let us consider the series $x_0, \ldots, x_{N-1}$ with

$$x_n = \sum_{i=1}^{r} f_{i,n}, \quad \text{where} \quad f_{i,n} = b_k \cos(\omega_i n + \gamma_i) \tag{16}$$

with pairwise different frequencies $\omega_i \in (0, 1/2)$ and amplitudes $|b_i|$ that satisfy the inequality $1 = b_1 > |b_2| > \ldots > |b_r| > 0$.

Let $N \to \infty$. For asymptotic separation of each term $f_{i,n}$ in sum (16), we apply the SSA method with the following parameters: the window width $L/N \to \alpha \in (0, 1)$; for reconstruction of the ith terms in sum (16), two principal components with numbers $2i - 1$ and $2i$ are used (see [1], Section 1).

Let us denote the reconstructed values by $\tilde{f}_{i,n}$ and set $r_{i,n}(N) = \tilde{f}_{i,n} - f_{i,n}$.

Theorem 3. *Let us use the denotation* $\delta_k = b_{k+1}/b_k$ *at* $1 \le k < r$. *If at all* k

$$|\delta_k| < 0.5 \frac{1}{1 + \sum_{j=k+2}^{r} \left(\prod_{i=k+1}^{j-1} \delta_i \right)^2}, \tag{17}$$

then $\max_{0 \le n < N} |r_{i,n}(N)| = O(N^{-1})$ *at* $1 \le i \le r$.

Proof. Let us denote for conciseness $\cos_j = \cos(2\pi\omega_j n + \gamma_j)$ and rewrite (16) in terms of δ_i.

$$x_n = \cos_1 + \delta_1 \cos_2 + \delta_1\delta_2 \cos_3 + \ldots + \delta_1 \ldots \delta_{r-1} \cos_r = \cos_1 + \sum_{j=2}^{r} \left(\prod_{i=1}^{j-1} \delta_i \right) \cos_j. \tag{18}$$

Fixing $1 \le k < r$, we write (18) in the form

$$x_n = \cos_1 + \sum_{j=2}^{k} \left(\prod_{i=1}^{j-1} \delta_i \right) \cos_j + \delta_k \left(\prod_{i=1}^{k-1} \delta_i \cos_{k+1} + \sum_{j=k+2}^{r} \left(\prod_{i=1}^{j-1} \delta_i \right) / \delta_k \cos_j \right) \tag{19}$$

and, with denotations $\beta_i = \prod_{i=1}^{j-1} \delta_i$ at $j \le k$ and

$$\beta_j = \left(\prod_{i=1}^{j-1} \delta_i \right) / \delta_k = \prod_{i=1}^{k-1} \delta_i \prod_{i=k+1}^{j-1} \delta_i = \beta_k \prod_{i=k+1}^{j-1} \delta_i$$

at $j \ge k$, we rewrite (19) in the standard form

$$x_n = \cos_1 + \sum_{j=2}^{k} \beta_j \cos_j + \delta_k \sum_{j=k+1}^{r} \beta_j \cos_j,$$

thereby coming to the results of Theorem 1: if

$$|\delta_k| < 0.5 \min \left(\frac{|\beta_{k+1}|}{|\beta_k|}, \frac{\beta_k^2}{\sum_{j=k+1}^{r} \beta_j^2} \right), \tag{20}$$

then the maximum errors of reconstruction of the signal $f_n = \cos_1 + \sum_{j=2}^{k} \beta_j \cos_j$ from the noise $\delta_k \sum_{j=k+1}^{r} \beta_j \cos_j$ will have the form $O(N^{-1})$.

Now, note that $|\beta_{k+1}| = |\beta_k|$ and at $j > k$

$$\beta_j = \left(\prod_{i=1}^{j-1}\delta_i\right)/\delta_k = \prod_{i=1}^{k-1}\delta_i\prod_{i=k+1}^{j-1}\delta_i = \beta_k\prod_{i=k+1}^{j-1}\delta_i,$$

we obtain that the condition (20) has the form (17).

Now, let (17) be satisfied at any k. Taking $k = 1$ and $k = r - 1$, we obtain that $\max_{0\le n<N}|r_{i,n}(N)| = O(N^{-1})$ at $i = 1$ and $i = r$; from the fact that (17) is fulfilled at $k = i - 1$ and $k = i$, it follows that $\max_{0\le n<N}|r_{i,n}(N)| = O(N^{-1})$ at $1 < i < r$.

□

5. CONCLUSIONS

Of course, the result of Theorem 3 gives only the sufficient conditions for separability of harmonics. Thus, at $r = 2$ (and $k = 1$), condition (17) transforms to $|\delta_1| = |b_2| < 0.5$, while the numerical experiments show that, in fact, the condition must be $|b_2| < 1$.

For $r = 3$, conditions (17) have the form $|b_2| < 0.5/(1 + \delta_2^2)$, $|\delta_2| = |b_3/b_2| < 0.5$, which is not fulfilled in the example presented in the Introduction, where $b_3/b_2 = 0.75$ and $b_2 = 0.8 > 0.5$. This is related to application of the general technique of perturbations of self-conjugated operators developed in [3]. It is possible that adaptation of this technique to solving the concrete problems of the SSA method would make it possible to weaken these sufficient conditions.

It should also be noted that the rate of convergence $\max_i |r_i(N)| = O(N^{-1})$ for the errors of reconstruction is apparently standard for solving the problems of separation of a signal from a sum with oscillating noise. At least, such a result is obtained in ([6], Section 2) for a growing exponential signal and a sinusoidal noise in the presence of signal discretization (as is shown in ([6], Section 1)); without discretization, $\max_i |r_i(N)|$ does not converge to zero at $N \to \infty$.

Similar estimates $\max_i |r_i(N)| = O(N^{-1})$ can be found in [8], where a linear signal is considered, as well as in [9], where the problems of the so-called recurrent forecast in the SSA are discussed.

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CONFLICT OF INTEREST

The author declares that he has no conflict of interest.

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