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# Algorithm for the Reduction of the Pareto Set Using a Collection of Fuzzy Information Quanta

**V. D. Noghin***

*Saint-Petersburg State University, Saint-Petersburg, Russia*

**e-mail: noghin@gmail.com*

**Abstract**—This paper deals with the multi-criteria problem of choice with a numerical vector function defined on a set of feasible variants. It is assumed that the decision maker uses a fuzzy preference relation for the choice process. Information on the preference relation is considered to be known in the form of a finite collection of fuzzy quanta. We formulate an algorithm to reduce the Pareto set in the multicriteria choice problem using the set of quanta in order to facilitate the final choice. A numerical example illustrates the operation of the algorithm.

**Keywords:** fuzzy sets, multicriteria choice, the Pareto set reduction, quanta of fuzzy information

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## INTRODUCTION

The problem of multi-criteria choice is to find, in the general case, the best subset of feasible variants, taking into account the given numerical criteria and fragmentary information on the preference relation of the decision maker (DM). Information here is in the form of information quanta. The presence of a quantum of information means that the DM is ready for a certain compromise, which consists in agreeing to accept some losses on less important criteria to obtain a certain gain on more important ones.

To solve the problem of multi-criteria choice, an axiomatic approach to the reducing of the Pareto set, developed by the author [1], was proposed and justified, over a long period. This approach admits that the DM's preference relation may be fuzzy; this kind of situation corresponds to a wide range of applied problems. Results have been obtained that make it possible to build an upper estimate for an unknown set of "best" options in the presence of certain quanta of fuzzy information. In this paper, one "gap" in the constructed axiomatic theory is filled; namely, an algorithm is proposed that indicates the means of constructing an upper estimate for the set of best variants using an *arbitrary* finite set of consistent information about the DM's fuzzy preferences. This algorithm can be considered as a development of the geometric algorithm for taking into account a set of quanta of crisp information, as earlier proposed by the author [2].

## THE PROBLEM OF FUZZY MULTI-CRITERIA CHOICE

Let on the universal set $U$ be the given fuzzy set $A$ with membership function $\lambda_A : U \to [0,1]$. A crisp set $\mathrm{supp}(X) = \{x \in A | \lambda_X(x) > 0\}$ called the carrier of the fuzzy set $A$. Further, we will use the standard operations of union, intersection, and addition of fuzzy sets [3, 4].

The fuzzy set defined on $R^m$ with membership function $\eta$ is called a cone if for any $x \in R^m$ and $\alpha > 0$ right $\eta(\alpha x) = \eta(x)$.

The fuzzy binary relation on a set $U$ with membership function $\mu(\cdot,\cdot)$ is called:

• transitive if $\mu(x,z) \geqq \min\{\mu(x,y);\mu(y,z)\}$ for all $x, y, z \in U$;

• ~~asymmetrical~~ if $\mu(x,y) > 0 \Rightarrow \mu(y,x) = 0$ for all $x, y \in U$;

• a cone relation in space $R^m$, if there is such a fuzzy cone $\eta : R^m \to [0,1]$, for which $\mu(x,y) = \eta(x-y)$ for all $x, y \in R^m$;

• invariant with respect to a linear positive transformation on $R^m$, if $\mu(\alpha x + c,\ \alpha y + c) = \mu(x,y)$ is true for all $x, y, c \in R^m$, and any $\alpha > 0$.

For an asymmetric binary relation with a membership function $\mu(\cdot,\cdot)$, defined on the set $Y \subset R^m$, the fuzzy set of non-dominated vectors $Ndom(Y)$ with membership function $\lambda_Y^N$ is defined by the formular [5]:

$$\lambda_Y^N(y) = 1 - \sup_{z \in Y} \mu(z, y) \text{ for all } y \in Y.$$

We denote a (crisp) set of feasible variants by $X$. Let there be $m$ numerical functions $f_1, f_2, \ldots, f_m$, given on the set $X$, with which the DM evaluates feasible variants. They form a vector criterion $f = (f_1, f_2, \ldots, f_m)$, which takes values in $m$-dimensional arithmetic vector space $R^m$. Each option $x \in X$ uniquely corresponds to some vector $y = f(x) \in R^m$ that characterizes this variant. The set of all feasible vectors is $Y = f(X)$.

We suppose that the DM is unable to unambiguously decide whether one of the two compared variants (vectors) is preferable to the other. This situation usually occurs when the compared variants (vectors) are evaluated using several criteria because for some of them one variant (vector) may turn out to be better than another, and for some others it may be worse. Nevertheless, often, based on their own preferences, DMs are able to choose one of these two compared variants (vectors), although his confidence in this choice may turn out to be less than 100%. In such cases, a mathematical apparatus based on a fuzzy preference relations is convenient.

For this reason, it is assumed that on the set of feasible variants $X$, an asymmetric fuzzy preference relation of the DM is given with the membership function $\mu_X(\cdot,\cdot)$. In the practice of solving applied problems, this relation is usually not completely known. Fragmentary information about it can be taken into account in the decision-making process in the form of fuzzy information quanta, which will be discussed below. For variants $x', x'' \in X$, number $\mu_X(x', x'') \in (0,1]$ interpreted as a degree of confidence that the variant $x'$ is preferable to $x''$ (i.e. the first variant dominates the second). In other words, out of two variants $x'$ and $x''$, the DM will choose the first and not choose the second with the degree of confidence $\mu_X(x', x'')$.

We list all the elements of the problem of fuzzy multi-criteria choice:

1) set of feasible variants $X$;

2) numerical vector criterion $f$, defined on the set $X$;

3) fuzzy preference relation $\succ_X$ with membership function $\mu_X(\cdot,\cdot)$ defined on the Cartesian product $X \times X$ and taking values in the interval $[0,1]$.

Obviously, when using fuzzy information on the preference relation of the DM, it is illogical in the general case to expect a crisp result upon completion of the solution of the choice problem. Therefore, as a solution to the problem of fuzzy multi-criteria choice, we consider some fuzzy subset of selected variants $C(X) \subset X$ and denote its membership function as $\lambda_X^C$. This set (this function) is to be found as a result of solving the choice problem.

The indicated multicriteria choice problem can also be formulated in terms of vectors. To this end, through $C(Y)$, we denote the fuzzy set of selectable vectors whose membership function naturally conjugates with the membership function of the fuzzy set of selectable variants:

$$\lambda_Y^C(y) = \begin{cases} \lambda_X^C(x), \text{ if } y = f(x) \text{ for some } x \in X; \\ 0, \text{ if } y \in R^m \setminus Y. \end{cases}$$

We assume that there is a one-to-one correspondence between the set of feasible variants and the set of corresponding vectors.

The membership function $\mu_Y(\cdot,\cdot)$ of fuzzy preference relation $\succ_Y$ on a set of vectors $Y$ is induced by means $\mu_X(\cdot,\cdot)$ in the following way:

$$\mu_Y(y, y') = \mu_X(x, x') \text{ for all } x, x' \in \tilde{X},$$

where $y = f(x)$, $y' = f(x')$ and $\tilde{X}$ is a set of equivalence classes on $X$ generated by the equality relation on $R^m$.

Each equivalence class consists of variants corresponding to the same feasible criteria vector. In turn, we can say that the membership function $\mu_Y(\cdot,\cdot)$ of the fuzzy preference relation $\succ_Y$ induces the membership function $\mu_X(\cdot,\cdot)$ of the fuzzy preference relation on the set $X$.

As a result, the problem of fuzzy multi-criteria choice in terms of vectors includes two objects: the set of feasible vectors $Y$ and the fuzzy preference relation with membership function $\mu_Y(\cdot,\cdot)$ defined on the set $Y$.

Thereby the problem of fuzzy multi-criteria choice is to find a fuzzy set of selectable vectors $C(Y) \subset Y$ with membership function $\mu_Y^C$.

## AN AXIOMATIC APPROACH TO REDUCING THE PARETO SET

We assume that the following four "reasonable" axioms [1] are fulfilled that impose certain restrictions on the behavior of DMs in the decision-making process.

**Axiom 1**. *For any pair of variants $x', x'' \in X$ for which* $\mu_X(x', x'') = \mu^* \in (0,1]$ we have $\lambda_X^C(x'') \leqq 1 - \mu^*$.

**Axiom 2.** *The fuzzy asymmetric preference relation $\succ_Y$ with the membership function $\mu_Y(\cdot,\cdot)$ (and hence also with the membership function $\mu_X(\cdot,\cdot)$) is transitive and, moreover, there is a transitive relation whose membership function we denote as $\mu(\cdot,\cdot)$, given over the whole space $R^m$ such that its restriction to $Y$ is the same as the preference relation $\mu_Y(\cdot,\cdot)$.*

They say that the criterion $f_i$ is compatible with preference relation $\mu(\cdot,\cdot)$, if for any vectors $y' = (y_1',\ldots,y_{i-1}',y_i',y_{i+1}',\ldots,y_m)$ and $y'' = (y_1',\ldots,y_{i-1}',y_i'',y_{i+1}',\ldots,y_m)$ from the fulfillment of the inequality $y_i' > y_i''$ the equality $\mu(y',y'') = 1$ follows.

**Axiom 3.** *Each of the criteria $f_1, f_2,\ldots, f_m$ is compatible with the preference relation $\mu(\cdot,\cdot)$.*

**Axiom 4.** *The fuzzy preference relation $\mu(\cdot,\cdot)$ is invariant with respect to a linear positive transformation.*

Axioms 1–4 identify a rather wide class of problems in which the DM's preference relation is, in the general case, partial. Thus, it is allowed to have pairs of incomparable variants (vectors) in this relation. It has been established [1] that under the conditions of fulfillment of Axioms 1–4, the preference relation $\mu(\cdot,\cdot)$ is conical with a acute sharp convex cone.

For vectors $a, b \in R^m$ we introduce a binary relation (Pareto relation):

$$a \geq b \Leftrightarrow a_i \geqq b_i \text{ for all } i, \quad a \neq b.$$

A set of Pareto optimal variants are defined by the equality:

$$P_f(X) = \{x^* \in X | \text{there is no } x \in X, \text{ such that } f(x) \geq f(x^*)\},$$

while the set of Pareto optimal vectors is given by the formula:

$$P(Y) = f(P_f(X)) = \{y^* \in Y | \text{there is no } y \in Y, \text{ such that } y \geq y^*)\}.$$

The membership function of the set of Pareto-optimal vectors (the characteristic function of this set) is denoted as follows:

$$\lambda_Y^P(y) = \begin{Bmatrix} 1, \text{ if } y \in P(Y); \\ 0, \text{ else.} \end{Bmatrix}$$

The properties of the Pareto set are discussed in detail in [6].

It follows from Axiom 1 that for any set of selected vectors $C(Y)$ takes place the inclusion $C(Y) \subset Ndom(Y)$. Thus, the fuzzy set of non-dominated vectors provides an upper estimate for the unknown set of selectable vectors. Moreover, the following result holds true.

**Fuzzy Edgeworth-Pareto Principle** [1]. *Let the Axioms 1–4 be fulfilled. Then for any fuzzy set of vectors $C(Y)$ is true the inclusion $C(Y) \subset P(Y)$, or, which is the same, the inequality $\lambda_Y^C(y) \leqq \lambda_Y^P(y)$ holds for all $y \in Y$.*

In accordance with this principle, the DM should choose the best vectors (variants) only within the Pareto set. This set is an upper estimate for $C(Y)$ (likewise for $C(X)$).

## Quanta of Fuzzy Information and Their Consistency

The formalization of information on the fuzzy preference relation for reducing the Pareto set is based on the following definition.

**Definition 1** [1]. We have a quantum of fuzzy information with a degree of confidence $\mu^* \in (0,1]$, if for some vector

$$y' \in N^m = \{y = (y_1, y_2,\ldots, y_m) \in R^m | y_i > 0, \; y_j < 0 \text{ for some } i, j \in I\}$$

is true $\mu(y', 0_m) = \mu^*$. Here $I = \{1,2,\ldots,m\}$.

For any vector $y' \in N^m$ there are two groups of criteria numbers $A$ and $B$ ($A, B \subset I, A \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$), such that $y_i' > 0$ for all $i \in A$, $y_j' < 0$ for all $j \in B$ and $y_s' = 0$ for all $s \in I \backslash (A \cup B)$.

In the case when there is a quantum of fuzzy information, we say that the group of criteria $A$ is more important than a group of criteria $B$ with parameters $w_i = y_i$, $w_j = -y_j$ for all $i \in A$, $j \in B$ and degree of confidence $\mu^*$. In the case where both sets $A$ and $B$ are singletons, i.e., $A = \{i\}$, $B = \{j\}$, they say that the $i$th criterion $f_i$ is more important than the $j$th criterion $f_j$.

The fact that for the DM the group of criteria $A$ is more important than the group $B$ implies the readiness of this DM to compromise, consisting in the fact that it agrees to lose no more than $w_j$ units according to less important group criteria $B$, expecting at the same time to acquire an increase of not less than $w_i$ units according to more important group criteria $A$. At the same time, the DM's propensity to compromise is equal to $\mu^* \times 100\%$. Parameter $\mu^* \in (0,1]$ adjusts the level of uncertainty associated with the specified trade-off: the larger this parameter, the smaller the amount of uncertainty. When $\mu^* = 1$, uncertainty as such disappears, which corresponds to the absolute readiness of the DM to this compromise.

For example, if $\mu((1,-2), 0_2) = 0.8$, then the first criterion is more important than the second, with parameters $w_1 = 1$, $w_2 = 2$ and degree of confidence 0.8.

In the particular case $\mu^* = 1$, we are dealing with a quantum of crisp information. Monograph [1] is devoted to studying the issues of taking into account such quanta in solving problems of multicriteria choice.

The following result shows how one fuzzy information quantum should be used to perform the reducing of the Pareto set.

**Theorem 1 [1]**. *Let a quantum of fuzzy information be given with criteria groups A, B, positive parameters* $w_i, w_j$ *for all* $i \in A$, $j \in B$ *and degree of confidence* $\mu^* \in (0,1]$. *Then for any set of selectable vectors* $C(Y)$ *with membership function* $\lambda_Y^C(\cdot)$ *we have*

$$\lambda_Y^C(y) \leqq \lambda^M(y) \leqq \lambda_Y^P(y) \text{ for all } y \in Y, \qquad (1)$$

*where* $\lambda^M(\cdot)$ *is the membership function, defined by the equalities:*

$$\lambda^M(y) = 1 - \sup_{z \in Y} \varsigma(z, y) \text{ for all } y \in Y, \qquad (2)$$

$$\varsigma(z, y) = \begin{cases} 1, \text{ if } z - y \in R_+^m, \\ \mu^*, \text{ if } \hat{z} - \hat{y} \in R_+^p, \ \ z - y \notin R_+^m, \\ 0, \textit{else}, \end{cases} \qquad (3)$$

$$\text{for all } y, \ z \in Y,$$

*and* $R_+^p = \{y \in R^p | y \geq 0_p\}$, $p = m - |B| + |A| \cdot |B|$, *and vector* $\hat{y}$ (*and* $\hat{z}$) *are made up of components* $y_i, i \in I \backslash B$ (*respectively*, $z_i, i \in I \backslash B$), *while the rest of the components have the form* $w_j y_i + w_i y_j$ (*respectively*, $w_j z_i + w_i z_j$) *for all* $i \in A$, $j \in B$.

Thanks to the axioms adopted above, the fuzzy preference relation with a membership function $\mu$ is conical, with a fuzzy acute convex $m$-dimensional cone [1]. The Pareto relation $\geq$ as well as fuzzy relation determined by the membership function (3) have similar properties. In addition, note that the fuzzy set with membership function (2), which, according to (1), gives the resulting upper estimate for the unknown set of selectable vectors $C(Y)$, is a fuzzy set of non-dominated vectors generated by relation (3).

Let a set of fuzzy information quanta be given, i.e., we have the set of vectors $u^i \in N^m$ along with the set of numbers $\mu_i \in (0,1]$, for which $\mu(u^i, 0_m) = \mu_i$, $i = 1, 2, \ldots, k$. Denote by $\mu_{11}, \ldots, \mu_{1k_1}$; $\mu_{21}, \ldots \mu_{2k_2}$; ...; $\mu_{l1}, \ldots, \mu_{lk_l}$ permutation of numbers $\mu_1, \mu_2, \ldots, \mu_k$, which represents them in descending order, i.e.

$$1 \geqq \mu_{11} = \ldots = \mu_{1k_1} > \mu_{21} = \ldots = \mu_{2k_2} > \ldots > \mu_{l1} = \ldots = \mu_{lk_l} > 0, \qquad (4)$$

where $k_1 + \ldots + k_l = k$, $1 \leqq l \leqq k$.

According to the introduced notation, the following one-to-one correspondence occurs: each vector $u^i$ corresponds to a certain positive number $\mu_{rs}$ ($r \in \{1, 2, \ldots, l\}$, $s \in \{1, 2, \ldots, k_r\}$), such that $\mu_i = \mu_{rs}$. Conversely, each given number $\mu_{rs}$ corresponds to some vector from the set $\{u^1, u^2, \ldots, u^k\}$.

Let $e^i$ be the unit vector of space $R^m$, $i = 1, 2, \ldots, m$. We introduce crisp cones $K_h$, $h = 1, 2, \ldots, l$, generated by unit vectors $e^1, e^2, \ldots, e^m$ along with all of those vectors $u^1, u^2, \ldots, u^k \in N^m$ that correspond to the numbers $\mu_i$ such that $\mu_i = \mu_{rs}$ with some $r$ and $s$, and $\mu_i \geqq \mu_{h1}$. Obviously, the introduced cones satisfy the inclusions $K_1 \subset K_2 \subset \ldots \subset K_l$.

The following definition introduces the concept of a consistent set of fuzzy information quanta.

**Definition 2** [1]. The v*ectors* $u^1, u^2, \ldots, u^k \in N^m$ *and the set of numbers* $\mu_1, \mu_2, \ldots, \mu_k \in (0,1]$ *define a consistent set of fuzzy information quanta*, *if there is such a fuzzy preference relation* $\mu(\cdot,\cdot)$, *satisfying the axioms* 1–4, *so that* $\mu(u^i, 0_m) = \mu_i$, $i = 1, 2, \ldots, k$.

Let $A$ be the numerical matrix of size $m \times n$. Let us agree that the homogeneous system of linear equations $Az = 0_m$ has $N$-solution if there is the vector $z^* \geq 0_n$, for which $Az^* = 0_m$.

**Theorem 2** [1]. *For the set of vectors* $u^1, u^2, \ldots, u^k \in N^m$ *along with a set of numbers* $\mu_1, \mu_2, \ldots, \mu_k \in (0,1]$ *define a consistent set of fuzzy information quanta if and only if the system of linear equations*

$$\lambda_1 e^1 + \ldots + \lambda_m e^m + \xi_1 u^1 + \ldots + \xi_k u^k = 0_m \qquad (5)$$

*relatively* $\lambda_1, \ldots, \lambda_m$, $\xi_1, \ldots, \xi_k$ *had no N-solutions and, further, each cone* $K_h$, $h = 1, \ldots, l-1$, *did not contain any vector* $u^i$, *such that* $\mu_i < \mu_{h1}$.

## AN ALGORITHM FOR CONSTRUCTING A DUAL CONE AND TAKING INTO ACCOUNT A SET OF QUANTA OF CRISP INFORMATION

Before moving on to the issue of the use of fuzzy information quanta to reduce the Pareto set, we present the information necessary for its consideration.

Let $a^1, a^2, \ldots, a^{m+k} \in R^m$ be a finite set of vectors. Denote the convex cone generated by the indicated vectors by means of $K = cone\{a^1, a^2, \ldots, a^{m+k}\}$. It is the collection of all non-negative linear combinations of vectors $a^1, a^2, \ldots, a^{m+k}$. We assume that this cone is acute, and its dimension is equal to $m$. Recall that the dimension of a cone is the same as the dimension of the minimal subspace containing the given cone.

The dual cone [7] with respect to the cone $K$ is denoted by the symbol $K^o$. It is defined by the equation:

$$K^o = \{x \in R^m | \langle x, y \rangle \geqq 0 \text{ for all } y \in K\}.$$

The dual cone for a polyhedral cone is also a polyhedral cone, which means that it is generated by some finite set of vectors. It is also known that the dual cone for an acute $m$-dimensional cone itself is acute and $m$-dimensional [7].

It was established [1] that to take into account an arbitrary finite set of quanta of crisp information, it is necessary to have an algorithm such that, for an arbitrary given finite set of vectors $a^1, a^2, \ldots, a^{m+k}$, generating an acute convex $m$-dimensional cone $K$, builds a minimal set of vectors, generating the dual cone $K^o$, i.e. vectors $b^1, b^2, \ldots, b^n$ such that $K^o = cone\{b^1, b^2, \ldots, b^n\}$. In geometric language, the problem is to construct the minimum set of normal interior vectors for all hyperplanes that are $(m-1)$-dimensional faces of the cone $K$ [8].

Having an algorithm to construct a dual cone, it is possible, for any finite consistent set of quanta of crisp information, to obtain formulas to recalculate the old vector criterion and form a new one, with the help of which an upper estimate is constructed for an unknown set of selected variants (vectors). We assume that the input of such an algorithm is a set of vectors $a^1, a^2, \ldots, a^{m+k}$, $k \geqq 1$, generating $m$-dimensional acute polyhedral cone in space $R^m$, and at the output (in memory) a new set of vectors $b^1, b^2, \ldots, b^n$ is formed to generate the dual cone in the same space. The justification of this algorithm can be found in [1]. Let us move on to its description.

**Step 1.** Open loop on variable $i$ from 1 to $C_{m+k}^{m-1}$ generating $m-1$ subsets from set vectors $a^1, a^2, \ldots, a^{m+k}$.

**Step 2.** If current $i$th subset $a^{i1}, a^{i2}, \ldots, a^{i(m-1)}$, selected from the set $a^1, a^2, \ldots, a^{m+k}$, is linearly dependent, then the number $i$ should be increased by one, and we return to the beginning of Step 2. When it is impossible to increase the number $i$ by 1, i.e., if $i = C_k^{m-1}$, we pass to Step 5. Otherwise, i.e. when the specified subset is linearly independent, perform Step 3.

**Step 3.** From subset of column-vectors $a^{i1}, a^{i2}, \ldots, a^{i(m-1)}$ form the square matrix $D$ of the $n$th order by assigning to the indicated columns on the right any of the vectors of the set $I_i = \{a^1, a^2, \ldots, a^{m+k}\} \backslash \{a^{i1}, a^{i2}, \ldots, a^{i(m-1)}\}$, which together with $a^{i1}, a^{i2}, \ldots, a^{i(m-1)}$ form linearly independent system. Compute the last column of inverse matrix $(D^T)^{-1}$, where $T$ is the symbol of transposition. This column vector (we denote it $\overline{y}^i$) should be remembered. By construction, vector $\overline{y}^i$ is orthogonal to all vectors of the subset $a^{i1}, a^{i2}, \ldots, a^{i(m-1)}$. Go to the next step.

**Step 4.** Calculate the products $\langle a^j, \overline{y}^i \rangle$ for all vectors $a^j \in I_i$. If at least one such product is negative, then the vector $\overline{y}^i$ is deleted from memory. In the case when all indicated scalar products are non-negative, increase the number $i$ by 1 and go to Step 2 (when such an increase is impossible, go to Step 5).

**Step 5.** Upon completion of the full cycle through the variable $i$, vectors that during the execution of the algorithm were written to memory as $\overline{y}^i$ are saved. They make up the desired minimum set of vectors $b^1, b^2, \ldots, b^n$, generating the dual cone $K^o$.

The following theorem shows how the described algorithm should be used to take into account an arbitrary finite set of crisp information quanta.

**Theorem 3** [1]. *Let the vectors $u^1, u^2, \ldots, u^k \in N^m$ generate a consistent set of information quanta. Then for any set of selectable vectors $C(Y)$,* the following *inclusions are valid:*

$$C(Y) \subset \hat{P}(Y) \subset P(Y), \tag{6}$$

*where $\hat{P}(Y) = f(P_g(X)) = P_g(Y)$, and the vector function*

$$g(x) = (\langle b^1, f(x) \rangle, \ldots, \langle b^n, f(x) \rangle) \quad (n \geqq m) \tag{7}$$

*built with using vectors $b^1, b^2, \ldots, b^n$, received by applying the algorithm for constructing the dual cone for the set of vectors $\{e^1, e^2, \ldots, e^m, u^1, u^2, \ldots, u^k\}$.*

According to Theorem 3, to use the set of crisp information quanta to reduce the Pareto set, it is necessary to apply the algorithm described above to the set of vectors $e^1, e^2, \ldots, e^m, u^1, u^2, \ldots, u^k$. Then, based on the vectors $b^1, b^2, \ldots, b^n$ found as a result of applying the algorithm, it is necessary to form a new vector criterion $g$ according to formula (7), the Pareto set with respect to which will form the upper estimate $\hat{P}(Y)$ for an unknown set of vectors $C(Y)$, taking into account the revealed set of information quanta.

## ALGORITHM FOR TAKING INTO ACCOUNT A SET OF FUZZY INFORMATION QUANTA

Let us proceed to the presentation of the key result of this work, namely, an algorithm making it possible to use the consistent fuzzy information available to the DM concerning the preference relation of the DM to find a new vector function $g$, which can be used to construct an upper estimate for the unknown set $C(Y)$.

Let the set of vectors $u^i \in N^m$ be given, along with a set of numbers $\mu_i \in (0,1]$, for which $\mu(u^i, 0_m) = \mu_i$, $i = 1, 2, \ldots, k$.

**Step 1.** Open loop on variable $h = 1,2,\ldots,l$. Find the Pareto set $Y_h = P(Y)$ and begin forming the desired fuzzy set with support $\operatorname{supp}(Y)$, assigning to all elements of the set $Y_h$ degree of membership equal to 1 and to the rest of its elements to 0.

**Step 2.** For the current cone $K_h$ $(h \in \{1,2,\ldots,l\})$ the set of its generators, together with the unit vectors of the space $R^m$, send to the input of the algorithm for constructing generators of the dual cone, and the vectors obtained as a result of the completion of this algorithm, using Theorem 3, are used to form the vector function $g^h$ and construct with its help a new current Pareto set $f(P_{g^h}(X))$. Go to the next step.

**Step 3.** Assign to all elements $Y_h \backslash f(P_{g^h}(X))$ membership degree $1 - \mu_{h1}$. If $h < l$, then put $h = h + 1$, $Y_h = f(P_{g^{h-1}}(X))$ and return to Step 2. Otherwise (i.e., when $h = l$) go to Step 4.

**Step 4.** End. As a result of the algorithm, each element of the set $\operatorname{supp}(Y)$ matches one of the numbers $0, 1, 1 - \mu_{h1}, 1 - \mu_{h2} \ldots, 1 - \mu_{hl}$. Thus, a fuzzy set will be formed, which is the desired upper estimate for the unknown set $C(Y)$, corresponding to a given consistent set of fuzzy information quanta.

This algorithm is finite, due to the finiteness of the set of information quanta and the finiteness of the algorithm for constructing generators of the dual cone. The validity of the algorithm follows from the fact that the cone of the fuzzy relation that provides the desired upper estimate for the unknown fuzzy set $C(Y)$ is the union of fuzzy cones supported by the cones $K_h$, and the value of the membership function for each such cone is constant and equal to $\mu_{h1}$. The algorithm is designed such that when it is applied, a fuzzy set of non-dominated vectors is sequentially constructed with respect to the specified fuzzy cone relation with the union of fuzzy cones described above. This set of non-dominated vectors, on the one hand, is included in the Pareto set, and on the other hand, it contains the arbitrary set $C(Y)$.

## ILLUSTRATIVE EXAMPLE

Let us demonstrate the described algorithm with the following example. Let

$$m = 3, \quad k = 3, \quad u^1 = (-2,3,1),$$
$$u^2 = (4,-1,1), \quad u^3 = (0,-3,2),$$

and $\mu(u^1, 0_3) = \mu(u^2, 0_3) = 0.8$, $\mu(u^3, 0_3) = 0.6$. At the same time, the set $Y = \{y^1, y^2, y^3, y^4\}$ consists of the following four vectors:

$$y^1 = (2,4,1), \quad y^2 = (2,3,1), \quad y^3 = (3,2,1),$$
$$\text{and} \quad y^4 = (4,1.5,1).$$

First, we check that the preference relation information given by the indicated three quanta is consistent. To do this, we compose a system of linear equations (5) for this case:

$$\lambda_1 - 2\xi_1 + 4\xi_2 = 0; \quad \lambda_2 + 3\xi_1 - \xi_2 - 3\xi_3 = 0;$$
$$\lambda_3 + \xi_1 + \xi_2 + 2\xi_3 = 0.$$

From the last equation, due to the non-negativity of the variables, it follows that: $\lambda_3 = \xi_1 = \xi_2 = \xi_3 = 0$. In this case, the first two equations imply $\lambda_1 = \lambda_2 = 0$. Thus, the system of equations (5) has no $N$-solutions.

Let us check the second condition of Theorem 3. Here, $l = 2$. Cone $K_1$ is generated by vectors $e^1, e^2, e^3, u^1, u^2$, and cone $K_2$ is generated by the vectors $e^1, e^2, e^3, u^1, u^2, u^3$. If cone $K_1$ contains a vector $u^3$, then there are non-negative and non-zero coefficients $\lambda_1, \lambda_2, \lambda_3, \xi_1, \xi_2$ under which the vector equality $\lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3 + \xi_1 u^1 + \xi_2 u^2 = u^3$ takes place; in coordinate form, they are the system of equalities:

$$\lambda_1 - 2\xi_1 + 4\xi_2 = 0; \quad \lambda_2 + 3\xi_1 - \xi_2 = -3;$$
$$\lambda_3 + \xi_1 + \xi_2 = 2.$$

Using the fact that the variables $\lambda_1, \lambda_2, \lambda_3$ are non-negative, we arrive at a system of inequalities:

$$-2\xi_1 + 4\xi_2 \leqq 0; \quad 3\xi_1 - \xi_2 \leqq -3; \quad \xi_1 + \xi_2 \leqq 2.$$

Adding the last two inequalities term by term, we find $4\xi_1 \leqq -1$, which contradicts to non-negativity of the variable $\xi_1$. Consequently, the second condition of Theorem 3 is also satisfied, which means that the given set of quanta is consistent.

Let us apply the algorithm described above to construct an upper estimate for an unknown fuzzy set of selectable vectors $C(Y)$.

We put $h = 1$. By virtue of $y^1 \geq y^2$, vector $y^2$ is not Pareto optimal. This is why $\lambda^M(y^2) = 0$, $Y_1 = P(Y) = Y \backslash \{y^2\}$ and all elements of the current Pareto set $Y_1$ get 1 as a degree of membership. In particular, $\lambda^M(y^4) = 1$.

In accordance with the algorithm for constructing a dual cone for $K_1$, a set of five vectors $\{e^1, e^2, e^3, u^1, u^2\}$ should be fed to its input, where $e^1 = (1,0,0)$, $e^2 = (0,1,0)$, $e^3 = (0,0,1)$. In this case, the cycle length is $C_5^2 = 10$.

We consider the first subset of two vectors $\{e^1, e^2\}$. Obviously, orthogonal to these vectors is, for example, the vector $e^3$, and $\langle e^3, u^1 \rangle = \langle e^3, u^2 \rangle = \langle e^3, e^3 \rangle = 1 > 0$. Therefore, the vector $\overline{y}^1 = e^3$ must be kept in memory.

Let us move on to the second subset $\{e^1, e^3\}$. Vector $e^2$ is orthogonal to both vectors of the considered subset, but $\langle e^2, u^2 \rangle = -1 < 0$. This means that the vector $e^2$ should be skipped. Now consider the pair $\{e^2, e^3\}$. Here, for the orthogonal vector $e^1$, $\langle e^1, u^1 \rangle = -2 < 0$. Therefore, this vector must also be skipped. For the set $\{e^1, u^1\}$ as an orthogonal vector, one can take, for example, $(0,1,-3)$. Because $\langle (0,1,-3), u^2 \rangle = -4 < 0$, the indicated orthogonal vector is again skipped.

For the set $\{e^2, u^1\}$ we can choose the orthogonal vector $\overline{y}^2 = (1,0,2)$, which, as it is easy to check, should be remembered. Further, we proceed similarly: for the set $\{e^3, u^1\}$; remember the orthogonal vector $\overline{y}^3 = (3,2,0)$, and for the set $\{e^1, u^2\}$ remember the orthogonal vector $\overline{y}^4 = (0,1,1)$. Consideration of other sets $\{e^2, u^2\}$, $\{e^3, u^2\}$, and $\{u^1, u^2\}$ will not lead to the appearance of additional vectors in the memory of the algorithm.

At the end of the loop, four vectors are stored in memory $\overline{y}^1, \overline{y}^2, \overline{y}^3, \overline{y}^4$. In accordance with Theorem 3, they correspond to the new vector criterion $g^1$ with components $g_{11}(y) = y_3$, $g_{12}(y) = y_1 + 2y_3$, $g_{13} = 3y_1 + 2y_2$, $g_{14} = y_2 + y_3$. Simple calculations show that

$$g^1(Y_1) = \{(1,10,14,5), (1,7,13,3), (1,7,15,2.5)\}.$$

In this set, the Pareto-optimal vectors are $y^1$ and $y^4$. According to the prescription at Step 2 of the algorithm, we set $\lambda^M(y^3) = 0.2$.

We increase $h$ by 1, i.e., $h = 2$ and introduce a new current Pareto set $Y_2 = Y_1 \backslash \{y^3\} = \{y^1, y^4\}$.

Now we feed vectors $e^1, e^2, e^3, u^1, u^2, u^3$ to the input of the algorithm construct the dual cone. Here, the vector $u^3$ was added to the previous set and the cycle length is now $C_6^2 = 15$.

To shorten the computation, we use the calculations already performed for $h = 1$. In addition to the completed cycle of length 10, it remains to check the signs of the scalar products of pairs of vectors in which $u^3$ is involved. We have $\langle \overline{y}^1, u^3 \rangle = 2 > 0$, $\langle \overline{y}^2, u^3 \rangle = 4 > 0$, but $\langle \overline{y}^3, u^3 \rangle = -6 < 0$. Therefore, the vector $\overline{y}^3$ should be removed from memory.

We continue the cycle. It remains to consider five options in which there is a vector $u^3$. We start with the pair $\{e^1, u^3\}$. For it, the orthogonal vector is, for example, $(0,-2,3)$. Checking the signs of scalar products, we find $\langle (0,-2,3), e^2 \rangle = -2 < 0$. Therefore, it should be skipped. We do the same with the other four pairs. As a result, we arrive at a single set $\{u^1, u^3\}$ with an orthogonal vector, to which we will assign the number of the previously deleted vector $\overline{y}^3 = (4.5,2,3)$. This needs to be remembered. The updated set of vectors $\overline{y}^1, \overline{y}^2, \overline{y}^3, \overline{y}^4$ is stored in memory; therefore, in accordance with Theorem 3, the new vector criterion takes the form:

$$g^2 = (y_3, y_1 + 2y_3, 4.5y_1 + 2y_2 + 3y_3, y_2 + y_3),$$

and $g^2(Y_2) = \{(1,4,14,20), (1,7,15,24)\}$.

In the resulting set, the second vector is Pareto optimal (i.e., $y^4$). One forms the final current Pareto set, therefore, $\lambda^M(y^1) = 0.4$.

As a result, the desired upper estimate for the set $C(Y)$ is a fuzzy set with a membership function:

$$\lambda^M(y^1) = 0.4, \quad \lambda^M(y^2) = 0,$$
$$\lambda^M(y^3) = 0.2, \quad \lambda^M(y^4) = 1.$$

As can be seen, the presence of fuzzy information quanta made it possible to significantly facilitate the choice for the DM. So, if the value of the degree of membership of the final decision is considered a priority, then based on the result obtained, the DM should stop his choice on the vector $y^4$. This constitutes the so-called a crisp part of the obtained upper estimate.

If, however, the indicated priority of the magnitude of the degree of membership of elements is abandoned, then at the final stage of decision-making, one can use, for example, the compromise approach outlined by the author in [9].

## CONCLUSIONS

To reduce the Pareto set through the use of fuzzy information quanta, a final algorithm is proposed and justified allowing an upper estimate to be built for an unknown fuzzy set of selectable vectors. One of its steps adopts the dual cone construction algorithm. An illustrative example shows that for problems that have a relatively small number of criteria and a finite set of possible vectors, this algorithm can be used manually, i.e., without using a computer.

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## CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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