ISSN 1547-4771, Physics of Particles and Nuclei Letters, 2023, Vol. 20, No. 5, pp. 1121–1123. © Pleiades Publishing, Ltd., 2023.
Russian Text © The Author(s), 2023, published in Pis'ma v Zhurnal Fizika Elementarnykh Chastits i Atomnogo Yadra, 2023.

PHYSICS OF SOLID STATE AND CONDENSED MATTER

Majorana Fermion as a Real Spinor Field

Yu. M. Pismak[a, *] and O. Yu. Shakhova[b, **]

[a] *Department of High Energy Physics and Elementary Particles, Faculty of Physics, St. Petersburg State University, St. Petersburg, 198504 Russia*

[b] *Naval Polytechnic Institute, Military Training and Research Center of the Naval Academy, St. Petersburg, 196604 Russia*

**e-mail: y.pismak@spbu.ru*

***e-mail: shahova_olga@outlook.com*

Received January 31, 2023; revised February 15, 2023; accepted March 14, 2023

Abstract—An analysis is made of the possibility of generalizing Majorana's theory based on the assumption of the reality of this spinor field using the general principles of relativistic physics. An action functional is constructed for it that contains two parameters: one has the dimension of mass and the other is dimensionless. Complete solutions of the Euler–Lagrange equations are found. An explicit form of the eigenspinors of the helicity operator is obtained, which can be used as basic elements in describing the states of the system. In it, in accordance with the results obtained by Majorana, the ordinary electric current is zero. The axial current is nontrivial. Its main features are studied and, within the framework of the modification of Majorana's theory, the possibility of participation of a real spinor fermionic field in electromagnetic interactions is discussed.

DOI: 10.1134/S154747712305062X

INTRODUCTION

The relativistic quantum field model of an elementary particle with spin $\frac{1}{2}$ suggested by Majorana [1] immediately attracted the attention of many researchers in the field of theoretical and experimental physics, and that has not faded throughout the time that has passed since its inception [2, 3]. Although there is still no direct experimental confirmation of the existence of Maiorano's fermions, interest in his theory has noticeably increased recently. The possibilities of the manifestation of the effects of these particles in various effects that are difficult to explain in any other way are being discussed in the field of elementary particle physics, nuclear physics, solid state physics, and cosmology [3–6]. In this paper, we consider the theory of a real spinor field without any comparison with Dirac's theory and using the consequences and analogies that follow from this and are presented in the article [1] and the works of many authors [2, 3]. Using only the reality of the fermion field and the main provisions of the relativistic theory as basic assumptions, we will carry out, to the extent possible, the most complete analysis of the physical properties of this field and consider the possibility of generalizing the results already known about Majorana fermions.

In the proposed [1] representation of gamma matrices

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma_1 = i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \tag{1}$$

where 0 is a 2×2 matrix with zero elements and σ_1, σ_2, σ_3 are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

all matrices (1) and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ are purely imaginary. We will be more comfortable with real $\overline{\gamma}^\mu = i\gamma^\mu$, $\mu = 0,1,2,3$, and $\overline{\gamma}^5 = i\gamma^5$. We assume that the model action in the coordinate representation should contain only real spinor fields, a differentiation operation no higher than the first order, and real parameters. It must be invariant under Lorentz transformations and have the form $\psi M^\mu_\nu \partial_\mu \psi + \psi M_{sc}\psi$. Here M^μ_ν are matrices for which $\psi M^\mu_\nu \psi$ is transformed as a vector under the Lorentz transformation of the spinor fields ψ, which ensures the invariance of the action under the corresponding transformations of all the quantities included in it. Due to the fermionicity of the spinor ψ, a nonzero contribution is made only by the part of the matrix that is symmetric with respect to the spinor indices of M^{μ}_{ν}.

Therefore, as four matrices M^{μ}_{ν}, we will use only symmetrical $M^{\mu}_{\nu} = M^{\mu T}_{\nu}$. Directly, the set of matrices $\overline{\gamma}^{\mu}$ by way of M^{μ}_{ν} is unsuitable due to antisymmetry $\overline{\gamma}_0$ and $\overline{\gamma}_2$. However, matrices $\overline{\gamma}^0\overline{\gamma}^{\mu}$ satisfy all the necessary requirements: they are real and symmetric and $\psi M^{\mu}_{\nu}\psi = \psi\overline{\gamma}^0\overline{\gamma}^{\mu}\psi$ is transformed as a vector under Lorentz transformations of spinors ψ. All matrices $\overline{\gamma}^0\overline{\gamma}^5\overline{\gamma}^{\mu}$ are antisymmetric, and this set of matrices is not suitable for constructing the kinetic contribution to the action.

Thus, the part of the Majorano fermion action containing derivatives has the form $(\psi\overline{\gamma}^0\overline{\gamma}^{\mu}\partial_{\mu}\psi - \partial_{\mu}\psi\overline{\gamma}^0\overline{\gamma}^{\mu}\psi)/2 = \psi\overline{\gamma}^0\overline{\gamma}^{\mu}\partial_{\mu}\psi$. To construct the contribution to the action without derivatives, we need antisymmetric real matrices M_{sc} for which $\psi M_{sc}\psi$ is invariant under Lorentz transformations of the field ψ. In the most general case, M_{sc} can be presented in the form $M_{sc} = m(\overline{\gamma}^0\cos(\zeta) + \overline{\gamma}^0\overline{\gamma}^5\sin(\zeta)) = M_{sc}(m,\zeta)$. Thus, the action for the Majorana field under our assumptions can be written as

$$\begin{aligned} S(\psi) &= \psi(\overline{\gamma}^0\overline{\gamma}^{\mu}\partial_{\mu} + M_{sc}(m,\zeta))\psi \\ &= \psi(\overline{\gamma}^0(\overline{\gamma}^{\mu}\partial_{\mu} + m(\mathbf{1}\cos(\zeta) + \overline{\gamma}^5\sin(\zeta))\psi. \end{aligned} \quad (2)$$

Here, $\mathbf{1}$ is a single 4×4-matrix m-mass dimension parameter and $0 \le \zeta < 2\pi$. The stationarity equation for action $S(\psi)$ (2) has the form

$$\overline{\gamma}^0(\overline{\gamma}^{\mu}\partial_{\mu} + m(1\cos(\zeta) + \overline{\gamma}^5\sin(\zeta))\psi = 0. \quad (3)$$

This is a linear homogeneous equation for the field ψ. The condition for its solvability is the equality to zero of the determinant of the matrix differential operator $L = \overline{\gamma}^0(\overline{\gamma}^{\mu}\partial_{\mu} + m(\mathbf{1}\cos(\zeta) + \overline{\gamma}^5\sin(\zeta))$: $\det(L) = (\partial^2 + m^2)^2 = 0$, where $\partial^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$. This means that each spinor component ψ satisfies the equation

$$(\partial^2 + m^2)\psi = 0. \quad (4)$$

The rank of the system of equations (3) is equal to two. Therefore, by solving two of them, we get the general solution. From the first two equations of (3), there are two linearly independent solutions in the form

$$\begin{aligned} \psi_1 = \{&(\partial_1 - m\sin(\zeta))\varphi_1, (m\cos(\zeta) - \partial_3)\varphi_1, \\ &(\partial_0 + \partial_2)\varphi_1, 0\}, \end{aligned} \quad (5)$$

$$\begin{aligned} \psi_2 = \{&(\partial_3 + m\cos(\zeta))\varphi_2, (\partial_1 + m\sin(\zeta))\varphi_2, \\ &0, -(\partial_0 + \partial_2)\varphi_2\}. \end{aligned} \quad (6)$$

Here, in curly brackets, there are four spinor components ψ_1 and ψ_2, expressed in terms of independent real fermionic functions $\varphi_1(x), \varphi_2(x)$, satisfying the equations (4) $L\varphi_1(x) = 0$, $L\varphi_2(x) = 0$. When substituting ψ_1, ψ_2 V (4), we get $\{0, 0, L\varphi_1, 0\} = 0$ and $\{0, 0, 0, L\varphi_2\} = 0$.

To describe the state of the field ψ, you can use the helicity operator

$$\Sigma = \frac{\overline{\gamma}_2\overline{\gamma}_3\partial_1 - \overline{\gamma}_1\overline{\gamma}_3\partial_2 + \overline{\gamma}_1\overline{\gamma}_2\partial_3}{2\sqrt{\Delta}},$$
$$\Delta = \partial_1^2 - \partial_2^2 - \partial_3^2,$$

as well as projectors $P_{\Sigma\pm} = 1/2 \pm \Sigma$ into a subspace with positive and negative helicity. Linear combinations $\psi_{\pm} = a_{\pm}\psi_1 + b_{\pm}\psi_2$ of spinors ψ_1, ψ_2 at

$$a_+ = \partial_0\partial_1 - \partial_2 m\sin(\zeta) - \sqrt{\Delta}(\partial_3 + m\cos(\zeta)),$$
$$b_+ = \partial_0\partial_3 + \partial_2 m\cos(\zeta) + \sqrt{\Delta}(\partial_1 - m\sin(\zeta)),$$
$$a_- = \partial_0\partial_1 - \partial_2 m\sin(\zeta) + \sqrt{\Delta}(\partial_3 + m\cos(\zeta)),$$
$$b_- = \partial_0\partial_3 + \partial_2 m\cos(\zeta) + \sqrt{\Delta}(m\sin(\zeta) - \partial_1)$$

are proper spinors of the helicity operator: $\Sigma\psi_{\pm} = \pm\frac{1}{2}\psi_{\pm}$.

Notice that Δ is a positive definite differential operator and the spinors $\psi_1, \psi_2, \psi_+, \psi_-$ in the coordinate representation, as well as the helicity operator Σ, are real. As part of our analysis, the presence in action $S(\psi)$, except for mass m additional dimensionless parameter $0 \le \zeta < 2\pi$, did not lead to a contradiction with our basic assumptions.

It is well known that the electric current vector $\psi\gamma^0\gamma^{\mu}\psi$ is equal to zero due to the symmetry of all matrices $\gamma^0\gamma^{\mu} = (\gamma^0\gamma^{\mu})^T$, $\mu = 0, 1, 2, 3$ in the Majorana representation (1) and fermionic field statistics ψ. Using (1), it is also easy to verify that the matrices $M^{\mu}_a = \gamma^0\gamma^5\gamma^{\mu}$ are antisymmetric: $M^{\mu T}_a = -M^{\mu}_a$; axial current $J^{\mu}_a(x) = \psi(x)M^{\mu}_a\psi(x)$ is a pseudovector in terms of Lorentz transformations and does not trivially vanish.

Due to the fermionic statistics of the field ψ, $\partial_{\mu}J^{\mu}_a(x) = 2\psi(x)M^{\mu}_a\partial_{\mu}\psi(x)$. Therefore, the functional $\int\psi(x)M^{\mu}_a\partial_{\mu}\psi(x)dx = \frac{1}{2}\int\partial_{\mu}J^{\mu}_a(x)dx$ of fields ψ is zero if we neglect the contribution of the boundaries of the integration region. Usually such functionals are not used as actions in models of physical systems. We also will not consider this possibility of modifying the kinetic part of the action of the Majorana field. Thus, it follows from our assumptions that the free theory of a real spinor fermion is determined by an action of the form (2).

Now we will consider the possibilities of constructing a functional that describes the interactions of the field ψ. If we impose the requirement of locality and renormalizability of the model in the spacetime dimension $d = 3+1$, then the interaction action cannot be a polynomial containing ψ to the extent $n > 2$. Then, if the interaction does not violate the Lorentz invariance, it can include the fields used in the relativistic theory of elementary particles. If we assume that the interaction action must be Hermitian, then the direct interaction of the Majorana fermion with other spinor fields of the Standard Model turns out to be inadmissible.

As a result, only Higgs scalar fields and gauge vector bosons remain as possible partners. All of them have the same dimension as the mass and derivative, and according to this criterion they can be participants in the interaction. Difficulties in realizing this possibility arise because of the gauge principle of interactions fundamental to the Standard Model. It would be preferable not to break it by introducing a spinor fermionic field ψ without adding some additional properties to it, such as color and aroma, as a result of which ψ would lose its sterility.

We will discuss the technical difficulties associated with this using the examples of problems that need to be solved when constructing the interaction of electromagnetic and Majorana fields. They are related to the reality and the gauge invariance of the field ψ. It cannot compensate for the gauge transformation of the vector potential A_μ of the electromagnetic field by analogy with charged Dirac fields in quantum electrodynamics. It is necessary to find a mechanism to replace these fields for the reduction of the nontrivial additive contribution of the gauge transformation, which enters linearly into the action of the field interaction A_μ.

The only possible local Lorentz invariant functional of fields A_μ and ψ that can be considered as a model of their interaction is written as $S_{int}(A,\psi) = \lambda A_\mu J_a^\mu(\psi)$, where λ is a dimensionless parameter. If for the field ψ the continuity equation $\partial_\mu J_a^\mu(\psi) = 0$ of the axial current holds, then, with the gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \rho$, interaction action does not change, since $S_{int}(A',\psi) = S_{int}(A,\psi) - \rho\lambda\partial_\mu J_a^\mu = S_{int}(A,\psi)$.

You can try to build an example of a field that satisfies this requirement ψ using the plane wave solution for the classical equation (3). Substitute in (5), (6) $\varphi_1(x,p) = \cos(px+a)u, \varphi_2(x,p) = \cos(px+b)v$, where $px = p_\mu x^\mu$, p_1, p_2, p_3 are the components of the wave impulse, $p_0 = \sqrt{\vec{p}^2 + m^2}$ is its energy, u and v are the independent of coordinates x fermions, and a, b are two arbitrary parameters.

As a result, we obtain the general solution $\psi = \psi_1 + \psi_2$ of Eq. (3) defined by the wave vector $\vec{p}$ and energy p_0. For it, the electric current $V^\mu = \psi\bar{\gamma}^0\bar{\gamma}^\mu\psi$ is zero due to $\bar{\gamma}^0\bar{\gamma}^\mu = (\bar{\gamma}^0\bar{\gamma}^\mu)^T$ and satisfies the continuity equation $\partial_\mu V^\mu = 0$ in a trivial way. Thus, due to $V_\mu = 0$, the usual mechanism of interaction between the fermionic field and the electromagnetic field turns out to be unrealizable. However, $\partial_\mu J_a^\mu = 0$ for axial current $J_a^\mu = -\psi\bar{\gamma}^5\bar{\gamma}^0\bar{\gamma}^\mu\psi$ is possible only with $m = 0$.

If the massive field of Majorana ψ is not a solution to the stationarity equation (3), but $\partial_\mu J_a^\mu(\psi) = 0$, then such fields ψ can interact with the electromagnetic field A_μ as virtual quantum ones without violating gauge invariance. If there are two Majorana fields, massless ψ directly interacting with the massive ψ' and the photon field A_μ, which interacts invariantly only with ψ, then in such a system the massive field ψ' can affect electromagnetic processes.

Just like the Dirac field, the Majorana field can interact with the material environment. Using methods similar to those used in [7], we can construct the corresponding interaction functionals.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. E. Majorana, "Teoria simmetrica dell'elettrone e del positrone," Nuovo Cimento **14**, 171—184 (1937).
2. S. M. Bilenky, "Neutrino Majorana," arXiv:hep-ph/0605172v1 (Accessed May 15, 2006).
3. F. Vissani, "What is matter according to particle physics and why try to observe its creation in lab," Universe **7**, 61 (2021).
4. C. M. Varma, "Majoranas in mixed-valence insulators," Phys. Rev. B **102**, 155145 (2020).
5. M. Z. Abyanth and M. Farhoudi, "Current density of Majorana bound states," Phys. Lett. A **453**, 128475 (2022).
6. M. Gomes, P. R. S. Gomes, K. Raimundo, R. C. B. Santos, and A. J. da Silva, "Topological superconductor from the quantum Hall phase: Effective field theory description," Phys. Rev **106**, 195111 (2022).
7. Yu. M. Pismak and O. Yu. Shakhova, "Singular background in a model of material plane interacting with Dirac particles," Phys. Part. Nucl. **53**, 326–334 (2022).