ZENO ERA AND NON-DECAYING SUBSPACES FOR MULTILEVEL FRIEDRICHS MODEL

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Exploring a possibility of using the Zeno effect for struggling decoherence, we study the short- and medium-time behavior of the survival probability of decaying states in the N-level Friedrichs model focusing on the degenerated and nearly degenerated systems. We show that in these systems decay can be considerably slowed down or even stopped by appropriate choice of initial conditions.

We consider the multilevel Friedrichs model described by the Hamiltonian:

$$H = H_0 + \lambda V, \tag{1}$$

where

$$H_0 = \sum_{k=1}^{N} \omega_k |k\rangle\langle k| + \int_0^\infty d\omega\, \omega |\omega\rangle\langle\omega|, \quad V = \sum_{k=1}^{N} \int_0^\infty d\omega f_k(\omega) \left(|k\rangle\langle\omega| + |\omega\rangle\langle k|\right). \tag{2}$$

Here $|k\rangle$ represent states of the discrete spectrum with the energy ω_k, $\omega_k > 0$. Degeneracy is reflected in ω_k. The vectors $|\omega\rangle$ represent states of the continuous spectrum with the energy ω, $f_k(\omega)$ are the form factors for the transitions between the discrete and the continuous spectrum, and λ is the coupling parameter. The Hamiltonian H_0 has continuous spectrum over the interval $[0, \infty)$ and discrete spectrum $\omega_1, ..., \omega_k$ embedded in the continuous spectrum. As the interaction λV is switched on, the discrete energy levels of H_0 become resonances of H as in the case of the one-level Friedrichs model[1]. As a result, the total evolution normally leads to the decay of any initial state in point spectrum eigenspace:

$$|\Phi\rangle = \sum_k \alpha_k |k\rangle, \quad \langle\Phi|\Phi\rangle = 1. \tag{3}$$

Decay is described by the survival probability $p(t)$

$$p(t) \equiv |\langle\Phi|e^{-iHt}|\Phi\rangle|^2 = |A(t)|^2, \tag{4}$$

where the survival amplitude $A(t)$ can be directly calculated in terms of the matrix elements of the partial resolvent $G(\omega)$ [2].

We here concentrate mainly on the model close to the completely degenerate one while the model without degeneracy has been discussed previously [2]. We suppose that the form factors can be expressed as

$$f_k(x) = f(x) + \varepsilon q_k(x). \tag{5}$$

We assume that ε is small, and consider the expansion in the vicinity of $\varepsilon = 0$. This choice is motivated by the expected similarity of the form factors for the degenerate levels. We investigate two different cases. In the first case, the form factors are identical, $\varepsilon = 0$, and in the second case they are different but similar, $\varepsilon \neq 0$ but ε is small.

For the first case $\varepsilon = 0$, the partial resolvent and its determinant are found explicitly. Using those expressions, we can show that when the energy levels x_k are well separated, each of them becomes a resonance z_k for nonzero λ^2. This case is discussed in detail in [2].

Let us now discuss not-separated levels. First, we consider the case when all energy levels are degenerated: $x_k = \bar{x}$ for any k. The matrix elements of the partial resolvent have two poles in the vicinity of $\bar{x}$: the real pole $z_1 = \bar{x}$ and the complex pole z_2 defined by the equation $\bar{x} - z_2 - N\lambda^2 W(z_2) = 0$, where $W(x) = \int_0^\infty dx' f^2(x')/(x' - x + i0)$. For small λ^2 we find

$$z_2 = \bar{x} - \lambda^2 N W(\bar{x}) + O(\lambda^4). \tag{6}$$

Let us find the time evolution of state $|\Phi\rangle$ (3). In our case, there exist only two poles z_1 and z_2 in a vicinity of the real axis. The survival amplitude $A(t)$ is

$$A(t) = e^{-i\bar{x}\Lambda t}\left(1 - \frac{1}{N}(1 - \frac{e^{-i(z_2-\bar{x})\Lambda t}}{1+\lambda^2 N W'(z_2)})|\hat{\alpha}|^2\right) + \frac{\lambda^2|\hat{\alpha}|^2}{2\pi i}\int_C dx \frac{W(x)e^{-ix\Lambda t}}{(\bar{x} - x - N\lambda^2 W(x))(\bar{x} - x)}, \tag{7}$$

where we have introduced $\hat{\alpha} = \sum_k \alpha_k$. We observe that for almost all initial states, the oscillations of the survival amplitude may not vanish with time. Indeed, if $\hat{\alpha} = 0$ then there is no decay at all and the survival probability is $p(t) = |A(t)|^2 = 1$. The survival probability decays to zero if and only if the initial state is $\alpha_k = e^{i\phi}/\sqrt{N}$ for any k and some real ϕ. For such states one has $|\hat{\alpha}|^2 = N$. For an arbitrary initial state, the decay is incomplete and

$$\lim_{t\to\infty} p(t) = \left(1 - |\hat{\alpha}|^2/N\right)^2 \neq 0.$$

We note that the decay defined by Eq. (7) is oscillating [2]. However, if the parameters of the quantum system are such that the system experiences complete decay (i.e. $p(\infty) = 0$) then the system decays without oscillations.

Let us now discuss the situation when the system is not completely degenerate, but it is close to the degenerate one. Namely, we consider the case when one energy level differs from others: $x_k = \bar{x}$ for $k = 1 \ldots N-1$, $x_N = \bar{x} + \Delta$. If the form factors are identical, there exist three different roots: the root $z_1 = \bar{x}$ with the multiplicity $N-2$ and the roots $z_{2,3}$ defined by

$$z_{2,3} = \bar{x} + \frac{\Delta - \lambda^2 N W(z) \mp \sqrt{(\Delta - \lambda^2 N W(z))^2 + 4\lambda^2\Delta(N-1)W(z)}}{2}. \tag{8}$$

Expression (8) gives the values for the roots for any Δ and λ^2. However, the limit when both these parameters go to zero is irregular. We will show that the pole and resolvent structure depends on the order, in which the limits are taken.

When λ^2 Re $W(\bar{x}) \ll \Delta$, we find

$$z_2 = \bar{x} - \lambda^2(N-1)W(\bar{x}) + O(\lambda^4), \quad z_3 = x_N - \lambda^2 W(x_N) + O(\lambda^4). \tag{9}$$

The root z_2 corresponds to the root (6) (the multiplicity is less by 1), and the root z_3 in this case is well-separated from $z_{1,2}$.

For the situation when $\Delta \ll \lambda^2$ Re $W(\bar{x})$, we find

$$z_2 = \bar{x} - \lambda^2 N W(\bar{x}) + O(\Delta), \quad z_3 = x_N - \frac{1}{N}\Delta + \frac{\Delta^2}{\lambda^2 W(\bar{x})}\frac{N-1}{N^3} + O(\Delta^3). \tag{10}$$

Again, the root z_2 corresponds to the root (6) and its imaginary part does not disappear when $\Delta \to 0$. The third root z_3 becomes real for identical energies, and the corresponding decay rate

$$\gamma_3 = -\frac{2\pi\Lambda f^2(\bar{x})\Delta^2}{\lambda^2|W(\bar{x})|^2}\frac{N-1}{N^3} \to 0 \quad \text{when} \quad \Delta \to 0.$$

The survival amplitude $A(t)$ is

$$\begin{aligned} A(t) \quad &\approx \quad e^{-i\bar{x}\Lambda t}\left(\sum_{k=1}^{N-1}|\alpha_k|^2 - \frac{1}{N-1}|\sum_{k=1}^{N-1}\alpha_k|^2 + \frac{1}{N}e^{-i(z_2-\bar{x})\Lambda t}|\hat{\alpha}|^2\right. \\ &+ \quad \left. e^{-i(z_3-\bar{x})\Lambda t}\left(\frac{1}{N-1}|\sum_{k=1}^{N-1}\alpha_k|^2 + |\alpha_N|^2 - \frac{1}{N}|\hat{\alpha}|^2\right)\right). \end{aligned} \tag{11}$$

This result reproduces formula (7) when $\Delta \to 0$. In our assumptions, we have two time scales for exponential decay: a fast decay defined by z_2 with decay rate proportional to $\lambda^2 N$, and a slow decay defined by z_3 with decay rate proportional to Δ^2. This slow decay is manifestation of non-degeneracy. The non-decaying subspaces of the system are now defined by two conditions: $|\hat{\alpha}| = 0$ and $\alpha_N = 0$.

In the case of different form factors, $\varepsilon \neq 0$, one cannot find a general explicit expression for the matrix elements of the partial resolvent G_{km} and its determinant. However, as the problem is the eigenvalue problem for a finite matrix, a general qualitative description is known (see e.g. Chapter II in [3]). Namely, for the system with identical energies $x_k \equiv \bar{x}$ in the vicinity of $\bar{x}$ there exist $N-1$ roots of the determinant, additionally to the root z_2 (6). Generally speaking, these roots give rise to exponentially decreasing terms in the survival amplitude $A(t)$. However, there may also exist real roots corresponding to bound states. These roots result in non-decaying behaviour of the survival probability.

Having in mind this qualitative description we shall analyze the pole structure of the resolvent by perturbation expansion. In the first non-vanishing order of the perturbation expansion with respect to ε, we calculate the following roots: $N-2$ roots of the type of $z_1 = \bar{x}$, the root z_2 (6) and the new root z_3. Its the imaginary part of z_3 can be calculated as

$$\mathrm{Im} z_3 = -\pi\varepsilon^2\lambda^2 \left[\sum_k q_k^2(\bar{x}) - \frac{1}{N}\left(\sum_k q_k(\bar{x}) \right)^2 \right] + O(\varepsilon^3\lambda^2). \tag{12}$$

One can see that $\mathrm{Im} z_3$ can be equal to zero even for different form factors $q_k(x)$. In this case, the decay will be slower, its width will be proportional to $\varepsilon^3\lambda^2$. It can be checked that $\mathrm{Im} z_3 = 0$ only for identical values $f_k(\bar{x})$. The values $f_k(\bar{x})$ define the widths of noninteracting resonances in the weak coupling limit. Therefore, for the resonances with the equal widths one has $\mathrm{Im} z_3 = 0$ up to $\varepsilon^3\lambda^2$ term. Then there exists a slowly decaying subspace.

In order to study the behaviour of the system (2) in the Zeno [4] and anti-Zeno [5] eras, we have used two approaches. One is based on the Taylor expansion of the survival probability [6] while another one based on the variable decay rate $\gamma(t)$ [7] defined as

$$p(t) = e^{-2\gamma(t)t}\,.$$

We have carefully analyzed the short time behaviour for a few different situations including the decay of one level, the completely degenerate case, and multilevel degenerate model with one different level.

We have considered the temporal behaviour of the survival probability in the multilevel Friedrichs model for degenerate and nearly degenerate situations. In the intermediate exponential era we have found a rich variety of behaviour ranging from pure exponential decay to exponentially decaying oscillations. For initial states belonging to the nondecaying subspaces, these oscillations stabilize without decaying to zero. The experimental implementation of this result should be exploited as a mean of suppression of decoherence.

In the short-time scale, our analysis has shown also a possibility for considerable slowing down of the decay due to the Zeno effect in the nearly degenerate system for a special class of initial conditions. If these systems and conditions are realizable experimentally (in atoms, ions, quantum dots etc.), one has a new possibility for efficient suppression of decoherence in quantum computation and communication using the Zeno effect. We have analyzed and compared two different definitions of the Zeno time.

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