Density of refinement masks for framelets

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The main result

Theorem

Suppose $f \in C(\mathbb{T})$, $f(0) = 1$, $|f(\xi)|^2 + |f(\xi + \pi)|^2 \leq 1$, $\varepsilon > 0$. Then there exists a compactly supported Parseval wavelet frame with a refinement mask m_0 such that $\|f - m_0\|_C < \varepsilon$. The refinable function φ has stable integer shifts.

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A Parseval wavelet frame

$$\psi_r \in L_2(\mathbb{R}), \quad r = 1, \dots, q, \quad \psi_{r,j,k}(x) := 2^{j/2}\psi_r(2^j x + k),$$

$$\sum_{r=1}^{q} \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{r,j,k}\rangle|^2 = \|f\|^2 \quad \text{for} \quad f \in L_2(\mathbb{R}).$$

Unitary extension principle (UEP, A.Ron, Zh.Shen, 1996)

Suppose there exist functions $\varphi \in L_2(\mathbb{R})$ (*the refinable function*) and $m_0 \in L_2(\mathbb{T})$ (*the refinement mask*) such that

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi),$$

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and $\lim_{\xi\to 0}\widehat{\varphi}(\xi) = 1$. We fix 2π-periodic functions $m_1, \dots, m_q \in L_2(\mathbb{T})$ (*the wavelet masks*) and define $\psi_1, \dots, \psi_q \in L_2(\mathbb{R})$ (*the wavelet generators*) by

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If for a.e. $\xi \in \mathbb{T}$

$$\begin{cases} \sum_{r=0}^{q} |m_r(\xi)|^2 = 1, \\ \sum_{r=0}^{q} m_r(\xi)\overline{m_r(\xi+\pi)} = 0, \end{cases}$$

then the functions $\{\psi_{r,j,k}\}_{j,k\in\mathbb{Z},r=1,\dots,q}$ form a Parseval frame for $L_2(\mathbb{R})$.

Motivation

$$\begin{cases} \sum\limits_{r=0}^{q} |m_r(\xi)|^2 = 1, \\ \sum\limits_{r=0}^{q} m_r(\xi)\overline{m_r(\xi+\pi)} = 0, \end{cases} \implies |m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \le 1.$$

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$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h(n)e^{in\xi}, \qquad (h(n)),\ n \in \mathbb{Z}, \text{ is a low-pass filter.}$$

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Approximating a function by a mask we can set a form of a low-pass filter.

Stability of integer shifts

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- A set of different complex numbers $\{b_1, \dots, b_n\}$ is called *cyclic* if $b_{j+1} = b_j^2$, $j = 1, \dots, n$, and $b_n^2 = b_1$. The cycle $\{1\}$ is called *trivial*.

Sufficient conditions for a mask

Remark

Let m_0 *be a trigonometric polynomial,* $m_0(0) = 1$*, and* $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \le 1$*. Assume* $\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j)$*. By the Mallat theorem* $\varphi \in L_2(\mathbb{R})$*, and since* $\widehat{\varphi}$ *is an entire function of exponential type, it follows that* $\widehat{\varphi}$ *is continuous at zero and* $\widehat{\varphi}(0) = m_0(0) = 1$*. Therefore, by UEP* m_0 *generates wavelet frame and wavelet generator has a compact support.*

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Assumptions on a mask

- m_0 is a trigonometric polynomial;
- $m_0(0) = 1$;
- $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \leq 1$;
- m_0 has neither nontrivial cycles nor a pair of symmetric roots on $\mathbb{T}$;
- $\|m_0 - f\|_C < \varepsilon$.

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2. We still have $|f_3(\xi)|^2 + |f_3(\xi+\pi)|^2 \leq 1$.
3. $j \in \mathbb{N}$, $k = 0, \dots, 2j-1$,

$$H_j(\xi) := \sum_{k=0}^{2j-1} f_3(\pi k/j) t_k^j(\xi), \quad \text{where} \quad t_k^j(\xi) := \left(\frac{\sin j\xi}{2j \sin \frac{\xi - \pi k/j}{2}} \right)^2 .$$

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H_j has neither nontrivial cycles nor a pair of symmetric roots on $\mathbb{T}$.

Sketch of proof

$$|H_j(\xi)|^2 + |H_j(\xi+\pi)|^2 = \left(\sum_{k=0}^{2j-1} f_3\left(\frac{\pi k}{j}\right) t_k^j(\xi)\right)^2 + \left(\sum_{k=0}^{2j-1} f_3\left(\frac{\pi k}{j}\right) t_{k+j}^j(\xi)\right)^2$$

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$$\leq \left(\sum_{k=0}^{2j-1} t_k^j(\xi)\right)^2 = 1. \qquad H_j = m_0.$$

A low-pass filter

The low-pass filter $(h(m))_{m=-2j+1}^{2j-1}$ corresponding the refinement mask H_j is

$$h(m) = \left(1 - \frac{|m|}{2j}\right) \tilde{f}_3(m),$$

where $\tilde{f}_3(m) = \frac{1}{2j} \sum_{k=0}^{2j-1} f_3\left(\frac{\pi k}{j}\right) e^{-im\frac{\pi k}{j}}$ is the inverse discrete Fourier transform of $\left(f_3\left(\frac{\pi k}{j}\right)\right)_{k=0}^{2j-1}$.

Higher approximation order?

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Since $H_j(\pi) = H_j'(\pi) = 0$, *it follows that the refinable function has the order of approximation* ≥ 2.

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Since $H_j(\pi) = H_j'(\pi) = 0$, it follows that the refinable function has the order of approximation ≥ 2.

Question. Is there an interpolation polynomial H_j such that

$$H_j(\xi) := \sum_{k=0}^{2j-1} f_3(\pi k/j) t_k^j(\xi), \qquad \text{where} \qquad t_k^j(\xi) > 0.$$

$$H_j(\pi k/j) = f_3(\pi k/j), \qquad H_j^{(n)}(\pi k/j) = 0, \quad n = 1, \dots, N,$$

$$\sum_{k=0}^{2j-1} t_k^j(\xi) = 1, \qquad \|H_j - f\|_C \to 0 \text{ as } j \to \infty,$$

$k = 0, \dots, 2j-1$?

Thank you for attention!