

# Splines of the Second and Seventh Order Approximation and the Stability of the Solution of the Fredholm Integral Equations of the Second Kind

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*Abstract:* - This work is a continuation of a series of works on the use of continuous local polynomial splines for solving interpolation problems and for solving the Fredholm integral equation of the second kind. Here the construction of a numerical solution to the Fredholm integral equation of the second kind using local spline approximations of the second order and the seventh order of approximation is considered. This paper is devoted to the investigation of the stability of the solution of the integral equation using these local splines. Approximation constants are given in the theorem about the error of approximation by the considered splines. Numerical examples of the application of spline approximations of the second and seventh order of approximation for solving integral equations are given.

*Key-Words:-* Fredholm integral equation of the second kind, splines of the seventh order of approximation, splines of the second order of approximation, stability, numerical solution.

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## 1 Introduction

Integral equations often arise in various applications. Many problems of astrophysics, mechanics, viscoelasticity, elasticity, vibrations, plasticity, hydrodynamics, electrodynamics, nuclear physics, biomechanics, geology, medicine problems, and many other problems are formulated in terms of integral equations. The mathematical model for many problems arising in different natural science industries is formulated using differential and integral equations. The investigation of these equations is conducted with the help of the numerical integration theory, [1]. Mathematics and physics problems are often reduced to solving integral or integro-differential equations. This is noted in the following papers. Hypoxia induced angiogenesis processes can be described by coupling an integro-differential kinetic equation of the Fokker-Planck type with a diffusion equation for the angiogenic factor, [2]. The charged particle motion for certain configurations of oscillating magnetic fields can be simulated by a Volterra integro-differential equation of the second order with time-periodic coefficients, [3]. In paper, [4], the Fourier integral transform has been employed to reduce the problem of determining the stress component under the contact region of a punch in

solving dual integral equations. In the paper, [5], the method of integral equations is proposed for some electrical engineering (current density, radiative heat transfer, heat conduction) problems. The presented models lead respectively to a system of Fredholm integral equations, integro-differential equations, or Volterra-Fredholm integral equations.

When solving integral equations, splines and wavelets are often used. The B-spline basis and the Hartree-Fock integro-differential equations are reduced to a computationally eigenvalue problem, [6]. In paper, [7], the Legendre wavelet functions were used to solve the Fredholm integral equation. In paper, [8], an efficient modification of the wavelets method to solve a new class of Fredholm integral equations of the second kind with a non-symmetric kernel is introduced. In paper, [9], the tension spline approximation to obtain the numerical solution of Volterra-Fredholm integral equation was developed. In paper, [10], a general spline maximum entropy method for the approximation of solutions for solving Fredholm integral equations was described. In paper, [11], the description of fuzzy Bezier splines is presented. An iterative numerical method for approximating the solution of fuzzy functional integral equations of the Fredholm type is proposed. In paper, [12], the non-polynomial

spline functions were used to obtain the numerical solutions of the Fredholm integral equations of the second kind. In paper, [13], the wavelet-Galerkin method for the numerical solution of the Fredholm linear integral equations and the second-order integro-differential equations are discussed. A construction of a quadratic spline-wavelet basis on the unit interval, such that the wavelets have three vanishing moments and the shortest support among such wavelets was proposed, [13]. In paper, [14], a new collocation technique for the numerical solution of the Fredholm, Volterra, and mixed Volterra-Fredholm integral equations of the second kind is introduced, and a numerical integration formula on the basis of the linear Legendre multi-wavelets is also developed. The linear Legendre multi-wavelets basis for the proposed method is used. In this technique, the unknown function is approximated by the truncated linear Legendre multi-wavelets series, [14].

Good results are obtained by using the Chebyshev polynomials. Paper, [15], focused on fuzzy Fredholm integral equations of the second kind. Using the Chebyshev polynomials due to their smoothness and reasonable behavior near boundaries, a new method is proposed to solve the fuzzy Fredholm integral equation. In paper, [16], the approximate solution of linear Fredholm integral equations of the second type on a closed interval is studied. The Galerkin method enhanced with the Chebyshev polynomials was used to improve the approximate solution.

We also note the following papers. In paper, [17], the authors have used the advanced multistep and hybrid methods to solve the Volterra integral equation. In paper, [18], the forward-jumping methods of the hybrid type are used for the construction of the methods with a high order of accuracy. In paper, [19], the Half-Sweep Gauss-Seidel iteration, which was used to find the approximate solution of the fuzzy Fredholm integral equations of the second kind, was applied. In paper, [20], linear Volterra-Fredholm integral equations of the second kind were considered in reproducing kernel space. A new scheme with a high convergence order for solving the approximate solutions to oscillation and non-oscillation of exact solutions was proposed.

In paper, [21], a new technique is offered to solve three types of linear integral equations of the 2nd kind, including the Volterra-Fredholm integral equations (as a general case), the Volterra integral equations, and the Fredholm integral equations (as special cases). The new technique depends on

approximating the solution to a polynomial of degree  $(m - 1)$  is described.

This work is a continuation of a series of works on the use of continuous local polynomial splines for solving interpolation problems and for solving integral equations, [22], [23]. This paper is devoted to the investigation of the stability of the solution of the integral equation using these local splines. As is known, the solution of integral equations of the second kind is reduced to finding the frame of the approximate solution. This means that we find approximations to the values of the function at the nodes of the grid (grid function). Usually, the integral equation is replaced by some difference scheme with a given order of accuracy. The approximate values of the function is converged to the values of the function at the grid nodes if the two conditions are fulfilled. These conditions are as follows: an approximation of the equation with a difference scheme, and the difference scheme is stable. Let for an approximate solution of the integral equation be

$$Au = f$$

a difference scheme is constructed

$$A_h u^h = f^h.$$

This scheme approximates the original equation with some order of accuracy. Suppose a linear normed space of functions defined on the grid is considered. The operator  $A_h$  maps the space  $U_h$  to the space  $F_h$ . A difference scheme is said to be stable on the right-hand side if, for any  $f^h \in F_h$ , the equation

$$A_h u^h = f^h$$

has a unique solution  $u^h \in U_h$  and

$$\| u^h \|_{U_h} \leq C \| f^h \|_{F_h}.$$

Here  $C$  is a constant. Next, consider the definition

$$\| u^h \|_{U_h} = \max_i |u^h(x_i)|.$$

Thus, the convergence of the approximate solution to the values of the function at the grid nodes follows from the approximation of the original equation and the stability of the solution.

In the works of, [24], conditions for the stability of the solution of the Fredholm integral equation of

the second kind using the trapezoidal method were obtained. Thus, the convergence of the obtained approximate solution of the Fredholm integral equation of the second kind to the values of the function at the grid nodes was proved.

Next, we construct a numerical scheme to solve the Fredholm integral equation of the second kind.

This paper discusses the stability of the solution when we use the local splines of the second and seventh order of approximation. We use these splines if the kernel and the right side are sufficiently smooth functions. To construct an approximate solution at the points between the grid nodes, we use the interpolation of the same local splines or the integral equation with an obtained solution at the nodes.

## 2 Problem Formulation

Let  $\{x_j\}$  be a grid of ordered nodes on the interval  $[a, b]$ :  $a = x_0 < \dots < x_n = b$ . Note that the approximations with the splines are constructed separately for each grid interval  $[x_j, x_{j+1}]$ . Let us assume that the values of the function  $u(x)$  are given at the grid nodes. The approximation using basis splines is built separately on each grid interval as the sum of the products of the values of the function  $u$  at the grid nodes and the basis splines  $\omega_j$ .

Let  $r, r_1, m$ , be integers,  $r + r_1 = m + 1$ ,  $r \geq 1, r_1 \geq 1$ , and the spline  $\omega_k$  be such that  $\text{supp } \omega_k = [x_{k-r}, x_{k+r_1}]$ . Following the methodology developed by Professor S.G. Mikhlin, we find the basis functions by solving the system of approximation relations

$$\sum_{j=k+1-r_1}^{k+r} x_j^s \omega_j(x) = x^s, \quad x \in [x_k, x_{k+1}],$$

$$s = 0, 1, \dots, m. \quad (1)$$

### 2.1 Polynomial Splines of the Second Order of Approximation

Let  $r = 1, r_1 = 1$ . The support of the basis splines of the second order of approximation occupies two grid intervals. These splines are convenient to use on a finite interval, both on a uniform grid of nodes and on a non-uniform grid of nodes. The approximation of the function on a finite interval of interpolation does not have a boundary layer. When solving the Fredholm integral equation of the second kind, the minimum number of the grid nodes is two. We set the support of the basis spline as follows:  $\text{supp } \omega_j = [x_{j-1}, x_{j+1}]$ . On the interval

$[x_j, x_{j+1}]$  we approximate the function  $u(x)$  by the following expression:

$$\tilde{u}(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x),$$

$$x \in [x_j, x_{j+1}],$$

where the basis splines  $\omega_j(x), \omega_{j+1}(x)$  are as follows:

$$\omega_j(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}}, \quad x \in [x_j, x_{j+1}],$$

$$\omega_{j+1}(x) = \frac{x - x_j}{x_{j+1} - x_j}, \quad x \in [x_j, x_{j+1}].$$

These splines are the interpolation splines of the second order of approximation as well as the first degree. The approximation using these splines is the continuous approximation.

Note that the minimum number of grid intervals is one, and the minimum number of grid nodes is two.

The following statement is valid for the approximation error.

Let  $h = x_{j+1} - x_j$ . In the case of the splines of the first degree, it is easy to obtain an estimate of the approximation error on the interval  $[x_j, x_{j+1}]$ , [22], [23]

$$|u(x) - \tilde{u}(x)| \leq \frac{h^2}{8} \max_{x \in [x_j, x_{j+1}]} |u''(x)|,$$

$$x \in [x_j, x_{j+1}].$$

In the case of an uneven grid of nodes, we take the length of the maximum grid interval as the value of  $h$ .

Further, we will use the norm of the form:

$$\|u\|_{C[a,b]} = \max_{x \in [a,b]} |u(x)|.$$

Consider the solution of the Fredholm integral equation of the second kind

$$(Au)(x) \equiv u(x) - \int_a^b K(x, s)u(s) ds = f(x),$$

$$x \in [a, b].$$

Here, and further, we assume that the kernel  $K(x, s)$  and the right side of the equation  $f(x)$  are

continuous. In addition, we assume that the equation is uniquely solvable and the estimate for the norm of the inverse operator in space  $C$  is known:

$$\|A^{-1}\| \leq B.$$

Suppose  $|K(x, s)| < \rho < 1$ , when  $0 \leq x \leq 1$ ,  $0 \leq s \leq 1$ .

We construct the set of nodes  $x_k, k = 0, 1, \dots, n$ , on the interval  $[a, b]$ . We have the relation

$$\int_a^b K(x, s)u(s) ds = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} K(x, s)u(s) ds.$$

On each interval  $[x_k, x_{k+1}]$  we replace  $u(s)$  with approximation  $\tilde{u}(x)$ .

Now we have the integral equation in the form:

$$u(x) - \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} K(x, s)(u_k \omega_k(s) + u_{k+1} \omega_{k+1}(s)) ds = f(x).$$

From here we get the system of equation in the form

$$u(x) - \sum_{k=1}^{n-1} (u_k A_k(x) + u_{k+1} A_{k+1}(x)) = f(x),$$

where

$$A_k(x) = \int_{x_k}^{x_{k+1}} K(x, s) \omega_k(s) ds,$$

$$A_{k+1}(x) = \int_{x_k}^{x_{k+1}} K(x, s) \omega_{k+1}(s) ds.$$

Next, we take  $x_j$  instead of  $x$  and we have to solve the system of linear algebraic equations:

$$u_j^h - \sum_{k=0}^{n-1} (u_k^h A_k(x_j) + u_{k+1}^h A_{k+1}(x_j)) = f_j^h,$$

$$j = 0, 1, 2, \dots, n - 1.$$

We assume that the integral  $\int_{x_k}^{x_{k+1}} K(x, s) \omega_k(s) ds$  can be computed exactly. Otherwise, we can use quadrature formulas.

Now, we suppose that  $a = 0, b = 1$ , and  $h = \text{const}$ . Let  $u_j$  be one of those components of the solution, whose absolute value is the largest. Therefore, the execution for this component of the solution is  $|u_j| = \max_k |u_k|$ .

We consider on the interval  $[x_k, x_{k+1}]$  the approximation of the function with splines of the second order of approximation

$$\tilde{u}(x) = u_k \omega_k(x) + u_{k+1} \omega_{k+1}(x).$$

Now we have

$$|f(x_j)| = |u(x_j) - \int_0^1 K(x_j, s) u(s) ds|,$$

where

$$\int_0^1 K(x_j, s) u(s) ds \approx \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} K(x_j, s)(u_k \omega_k(s) + u_{k+1} \omega_{k+1}(s)) ds.$$

We assume that  $h = x_{k+1} - x_k$ . It is easy to calculate the integrals  $\int_{x_k}^{x_{k+1}} \omega_k(s) ds = h/2$ ,  $\int_{x_k}^{x_{k+1}} \omega_{k+1}(s) ds = h/2$ .

Now using the mean value theorem of integral calculus, we obtain

$$\int_{x_k}^{x_{k+1}} K(x_j, s) \omega_k(s) ds = K(x_j, \eta_k) \int_{x_k}^{x_{k+1}} \omega_k(s) ds = K(x_j, \eta_k) \frac{h}{2}, \eta_k \in [x_k, x_{k+1}].$$

Similarly, we get

$$\int_{x_k}^{x_{k+1}} K(x_j, s) \omega_{k+1}(s) ds = K(x_j, \xi_k) \frac{h}{2}, \xi_k \in [x_k, x_{k+1}].$$

Finally, we obtain the inequality

$$|f(x_j)| \geq (1 - \rho) |u(x_j)|.$$

Thus, we have the estimation

$$|u(x_j)| \leq \frac{1}{1 - \rho} |f(x_j)|.$$

In particular, we can take  $f = 0$ . This implies that the system is uniquely solvable. The last inequality means the stability of the solution depends on the right side of the equation with the constant  $C = \frac{1}{1 - \rho}$ .

## 2.2 Polynomial Splines of the Seventh Order of Approximation

Now we consider the application of splines of the seventh order of approximation to solve the Fredholm integral equations of the second kind. Different modifications of the splines of the seventh order of approximation are used at the beginning, in the middle and at the end of the interpolation interval  $[a, b]$ . The support of the basis spline occupies seven grid intervals.

First, consider the approximation properties of polynomial splines of the seventh order of approximation.

Let  $r, r_1$  be integers,  $r + r_1 = 7, r \geq 1, r_1 \geq 1$ , and the spline  $\omega_k$  be such that  $\text{supp } \omega_k =$

$[x_{k-r}, x_{k+r_1}]$ . We find the basis functions by solving the system of approximation relations

$$\sum_{j=k+1-r_1}^{k+r} x_j^s \omega_j(x) = x^s, \quad x \in [x_k, x_{k+1}],$$

$$s = 0, 1, \dots, 6.$$

With different values of the parameters  $r, r_1$ , we get basis splines suitable for the approximation at the beginning of the interpolation interval (the right basis splines), in the middle of the interpolation interval (the middle basis splines), or at the end of the interpolation interval (the left basis splines).

With  $r_1 = 4$  and  $r = 3$  we get the middle splines. On the interval  $[x_k, x_{k+1}]$ , we construct the approximation with the middle splines at a distance of three grid intervals from the ends of the interval  $[a, b]$  in the form:

$$\tilde{u}^M(x) = \sum_{j=k-3}^{k+3} u(x_j) \omega_j^M(x), \quad x \in [x_k, x_{k+1}],$$

where the middle basis splines  $\omega_j^M(x)$  have the form:

$$\omega_{k-3}^M(x) = c_{k-3}(x)/d_{k-3},$$

where

$$c_{k-3}(x) = (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}) \times (x - x_k)(x - x_{k-1})(x - x_{k-2}),$$

$$d_{k-3} = (x_{k-3} - x_{k+3})(x_{k-3} - x_{k+2}) \times (x_{k-3} - x_{k+1})(x_{k-3} - x_k) \times (x_{k-3} - x_{k-1})(x_{k-3} - x_{k-2});$$

$$\omega_{k-2}^M(x) = \frac{c_{k-2}(x)}{d_{k-2}},$$

$$c_{k-2}(x) = (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}) \times (x - x_k)(x - x_{k-1})(x - x_{k-3}),$$

$$d_{k-2} = (x_{k-2} - x_{k+3})(x_{k-2} - x_{k+2}) \times (x_{k-2} - x_{k+1})(x_{k-2} - x_k) \times (x_{k-2} - x_{k-1})(x_{k-2} - x_{k-3});$$

$$\omega_{k-1}^M(x) = \frac{c_{k-1}(x)}{d_{k-1}},$$

$$c_{k-1}(x) = (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}) \times (x - x_k)(x - x_{k-2})(x - x_{k-3}),$$

$$d_{k-1} = (x_{k-1} - x_{k+3})(x_{k-1} - x_{k+2}) \times (x_{k-1} - x_{k+1})(x_{k-1} - x_k) \times (x_{k-1} - x_{k-2})(x_{k-1} - x_{k-3});$$

$$\omega_k^M(x) = \frac{c_k(x)}{d_k},$$

$$c_k(x) = (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}) \times (x - x_{k-1})(x - x_{k-2})(x - x_{k-3}),$$

$$d_k = (x_k - x_{k+3})(x_k - x_{k+2})(x_k - x_{k+1})$$

$$\times (x_k - x_{k-1})(x_k - x_{k-2})(x_k - x_{k-3});$$

$$\omega_{k+1}^M(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$c_{k+1}(x) = (x - x_{k+3})(x - x_{k+2})(x - x_k) \times (x - x_{k-1})(x - x_{k-2})(x - x_{k-3}),$$

$$d_{k+1} = (x_{k+1} - x_{k+3})(x_{k+1} - x_{k+2}) \times (x_{k+1} - x_k)(x_{k+1} - x_{k-1}) \times (x_{k+1} - x_{k-2})(x_{k+1} - x_{k-3});$$

$$\omega_{k+2}^M(x) = \frac{c_{k+2}(x)}{d_{k+2}},$$

$$c_{k+2}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-3})(x - x_{k+3}),$$

$$d_{k+2} = (x_{k+2} - x_{k+1})(x_{k+2} - x_k) \times (x_{k+2} - x_{k-1})(x_{k+2} - x_{k-2}) \times (x_{k+2} - x_{k-3})(x_{k+2} - x_{k+3}).$$

Approximations with these basis splines can be constructed on the grid intervals  $[x_k, x_{k+1}]$ ,  $k = 3, \dots, n - 3$ .

Let us consider the approximation with the left basis splines. We get the left basis splines when  $r_1 = 6, r = 1$ . In this case, formula (1) on the interval  $[x_k, x_{k+1}]$  takes the form:

$$\tilde{u}^L(x) = \sum_{j=k-5}^{k+1} u(x_j) \omega_j^L(x), \quad x \in [x_k, x_{k+1}].$$

where the basis splines  $\omega_k^L(x)$  have the form

$$\omega_{k-5}^L(x) = c_{k-5}(x)/d_{k-5},$$

$$c_{k-5}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-3})(x - x_{k-4}),$$

$$d_{k-5} = (x_{k-5} - x_{k+1})(x_{k-5} - x_k) \times (x_{k-5} - x_{k-1})(x_{k-5} - x_{k-2}) \times (x_{k-5} - x_{k-3})(x_{k-5} - x_{k-4});$$

$$\omega_{k-4}^L(x) = \frac{c_{k-4}(x)}{d_{k-4}},$$

$$c_{k-4}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-3})(x - x_{k-5}),$$

$$d_{k-4} = (x_{k-4} - x_{k+1})(x_{k-4} - x_k) \times (x_{k-4} - x_{k-1})(x_{k-4} - x_{k-2}) \times (x_{k-4} - x_{k-3})(x_{k-4} - x_{k-5});$$

$$\omega_{k-3}^L(x) = \frac{c_{k-3}(x)}{d_{k-3}},$$

$$c_{k-3}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-4})(x - x_{k-5}),$$

$$d_{k-3} = (x_{k-3} - x_{k+1})(x_{k-3} - x_k)$$

$$\begin{aligned} &\times (x_{k-3} - x_{k-1})(x_{k-3} - x_{k-2}) \\ &\times (x_{k-3} - x_{k-4})(x_{k-3} - x_{k-5}); \end{aligned}$$

$$\omega_{k-2}^L(x) = \frac{c_{k-2}(x)}{d_{k-2}},$$

$$\begin{aligned} c_{k-2}(x) &= (x - x_{k+1})(x - x_k)(x - x_{k-1}) \\ &\times (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}), \\ d_{k-2} &= (x_{k-2} - x_{k+1})(x_{k-2} - x_k) \\ &\times (x_{k-2} - x_{k-1})(x_{k-2} - x_{k-3}) \\ &\times (x_{k-2} - x_{k-4})(x_{k-2} - x_{k-5}); \end{aligned}$$

$$\omega_{k-1}^L(x) = \frac{c_{k-1}(x)}{d_{k-1}},$$

$$\begin{aligned} c_{k-1}(x) &= (x - x_{k+1})(x - x_k)(x - x_{k-2}) \\ &\times (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}), \\ d_{k-1} &= (x_{k-1} - x_{k+1})(x_{k-1} - x_k) \\ &\times (x_{k-1} - x_{k-2})(x_{k-1} - x_{k-3}) \\ &\times (x_{k-1} - x_{k-4})(x_{k-1} - x_{k-5}); \end{aligned}$$

$$\omega_k^L(x) = \frac{c_k(x)}{d_k},$$

$$\begin{aligned} c_k(x) &= (x - x_{k+1})(x - x_{k-1})(x - x_{k-2}) \\ &(x - x_{k-3})(x - x_{k-4})(x - x_{k-5}), \\ d_k &= (x_k - x_{k+1})(x_k - x_{k-1})(x_k - x_{k-2}) \\ &(x_k - x_{k-3})(x_k - x_{k-4})(x_k - x_{k-5}); \end{aligned}$$

$$\omega_{k+1}^L(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$\begin{aligned} c_{k+1}(x) &= (x - x_k)(x - x_{k-1})(x - x_{k-2}) \\ &\times (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}), \\ d_{k+1} &= (x_{k+1} - x_k)(x_{k+1} - x_{k-1}) \\ &\times (x_{k+1} - x_{k-2})(x_{k+1} - x_{k-3}) \\ &\times (x_{k+1} - x_{k-4})(x_{k+1} - x_{k-5}). \end{aligned}$$

Approximations with these basis splines can be applied on the next grid interval  $[x_k, x_{k+1}]$ ,  $k = 5, \dots, n - 1$ .

Let us consider the approximation with the left-right basis splines. We get the left basis splines when  $r_1 = 5$ ,  $r = 2$ . In this case, formula (1) on the interval  $[x_k, x_{k+1}]$  takes the form

$$\tilde{u}^{LR}(x) = \sum_{j=k-4}^{k+2} u(x_j)\omega_j^{LR}(x), x \in [x_k, x_{k+1}],$$

where the basis splines  $\omega_k^{LR}(x)$  have the form

$$\omega_{k-4}^{LR}(x) = c_{k-4}(x)/d_{k-4},$$

$$\begin{aligned} c_{k-4}(x) &= (x - x_{k+2})(x - x_{k+1})(x - x_k) \\ &\times (x - x_{k-1})(x - x_{k-2})(x - x_{k-3}), \\ d_{k-4} &= (x_{k-4} - x_{k+2})(x_{k-4} - x_{k+1}) \end{aligned}$$

$$\begin{aligned} &\times (x_{k-4} - x_k)(x_{k-4} - x_{k-1}) \\ &\times (x_{k-4} - x_{k-2})(x_{k-4} - x_{k-3}); \end{aligned}$$

$$\omega_{k-3}^{LR}(x) = \frac{c_{k-3}(x)}{d_{k-3}},$$

$$\begin{aligned} c_{k-3}(x) &= (x - x_{k+2})(x - x_{k+1})(x - x_k) \\ &\times (x - x_{k-1})(x - x_{k-2})(x - x_{k-4}), \\ d_{k-3} &= (x_{k-3} - x_{k+2})(x_{k-3} - x_{k+1}) \\ &\times (x_{k-3} - x_k)(x_{k-3} - x_{k-1}) \\ &\times (x_{k-3} - x_{k-2})(x_{k-3} - x_{k-4}); \end{aligned}$$

$$\omega_{k-2}^{LR}(x) = \frac{c_{k-2}(x)}{d_{k-2}},$$

$$\begin{aligned} c_{k-2}(x) &= (x - x_{k+2})(x - x_{k+1})(x - x_k) \\ &\times (x - x_{k-1})(x - x_{k-3})(x - x_{k-4}), \\ d_{k-2} &= (x_{k-2} - x_{k+2})(x_{k-2} - x_{k+1}) \\ &\times (x_{k-2} - x_k)(x_{k-2} - x_{k-1}) \\ &\times (x_{k-2} - x_{k-3})(x_{k-2} - x_{k-4}); \end{aligned}$$

$$\omega_{k-1}^{LR}(x) = \frac{c_{k-1}(x)}{d_{k-1}},$$

$$\begin{aligned} c_{k-1}(x) &= (x - x_{k+2})(x - x_{k+1})(x - x_k) \\ &\times (x - x_{k-2})(x - x_{k-3})(x - x_{k-4}), \\ d_{k-1} &= (x_{k-1} - x_{k+2})(x_{k-1} - x_{k+1}) \\ &\times (x_{k-1} - x_k)(x_{k-1} - x_{k-2}) \\ &\times (x_{k-1} - x_{k-3})(x_{k-1} - x_{k-4}); \end{aligned}$$

$$\omega_k^{LR}(x) = \frac{c_k(x)}{d_k},$$

$$\begin{aligned} c_k(x) &= (x - x_{k+2})(x - x_{k+1})(x - x_{k-1}) \\ &\times (x - x_{k-2})(x - x_{k-3})(x - x_{k-4}), \\ d_k &= (x_k - x_{k+2})(x_k - x_{k+1}) \\ &\times (x_k - x_{k-1})(x_k - x_{k-2}) \\ &\times (x_k - x_{k-3})(x_k - x_{k-4}); \end{aligned}$$

$$\omega_{k+1}^{LR}(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$\begin{aligned} c_{k+1}(x) &= (x - x_{k+2})(x - x_k)(x - x_{k-1}) \\ &\times (x - x_{k-2})(x - x_{k-3})(x - x_{k-4}), \\ d_{k+1} &= (x_{k+1} - x_{k+2})(x_{k+1} - x_k) \\ &\times (x_{k+1} - x_{k-1})(x_{k+1} - x_{k-2}) \\ &\times (x_{k+1} - x_{k-3})(x_{k+1} - x_{k-4}); \end{aligned}$$

$$\omega_{k+2}^{LR}(x) = \frac{c_{k+2}(x)}{d_{k+2}},$$

$$\begin{aligned} c_{k+2}(x) &= (x - x_{k+1})(x - x_k)(x - x_{k-1}) \\ &\times (x - x_{k-2})(x - x_{k-3})(x - x_{k-4}), \\ d_{k+2} &= (x_{k+2} - x_{k-1})(x_{k+2} - x_k) \\ &\times (x_{k+2} - x_{k-1})(x_{k+2} - x_{k-2}) \\ &\times (x_{k+2} - x_{k-3})(x_{k+2} - x_{k-4}). \end{aligned}$$

Approximations with these basis splines can be applied on the next grid intervals  $[x_k, x_{k+1}]$ ,  $k = 4, \dots, n - 2$ .

Consider the approximation with the right basis splines. Let  $r_1 = 1$ ,  $r = 6$ . In this case, formula (1) takes the next form on the interval  $[x_k, x_{k+1}]$ :

$$\tilde{u}^R(x) = \sum_{j=k}^{k+6} u(x_j) \omega_j^R(x), \quad x \in [x_k, x_{k+1}],$$

where the right basis splines  $\omega_j^R(x)$  have the form:

$$\omega_k^R(x) = \frac{c_k(x)}{d_k},$$

$$c_k(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}),$$

$$d_k = (x_k - x_{k+6})(x_k - x_{k+5})(x_k - x_{k+4}) \times (x_k - x_{k+3})(x_k - x_{k+2})(x_k - x_{k+1});$$

$$\omega_{k+1}^R(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$c_{k+1}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+3})(x - x_{k+2})(x - x_k),$$

$$d_{k+1} = (x_{k+1} - x_{k+6})(x_{k+1} - x_{k+5}) \times (x_{k+1} - x_{k+4})(x_{k+1} - x_{k+3}) \times (x_{k+1} - x_{k+2})(x_{k+1} - x_k);$$

$$\omega_{k+2}^R(x) = \frac{c_{k+2}(x)}{d_{k+2}},$$

$$c_{k+2}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+3})(x - x_{k+1})(x - x_k),$$

$$d_{k+2} = (x_{k+2} - x_{k+6})(x_{k+2} - x_{k+5}) \times (x_{k+2} - x_{k+4})(x_{k+2} - x_{k+3}) \times (x_{k+2} - x_{k+1})(x_{k+2} - x_k);$$

$$\omega_{k+3}^R(x) = \frac{c_{k+3}(x)}{d_{k+3}},$$

$$c_{k+3}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+3} = (x_{k+3} - x_{k+6})(x_{k+3} - x_{k+5}) \times (x_{k+3} - x_{k+4})(x_{k+3} - x_{k+2}) \times (x_{k+3} - x_{k+1})(x_{k+3} - x_k);$$

$$\omega_{k+4}^R(x) = \frac{c_{k+4}(x)}{d_{k+4}},$$

$$c_{k+4}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+3}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+4} = (x_{k+4} - x_{k+6})(x_{k+4} - x_{k+5}) \times (x_{k+4} - x_{k+3})(x_{k+4} - x_{k+2}) \times (x_{k+4} - x_{k+1})(x_{k+4} - x_k);$$

$$\omega_{k+5}^R(x) = \frac{c_{k+5}(x)}{d_{k+5}},$$

$$c_{k+5}(x) = (x - x_{k+6})(x - x_{k+4})(x - x_{k+3}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+5} = (x_{k+5} - x_{k+6})(x_{k+5} - x_{k+4}) \times (x_{k+5} - x_{k+3})(x_{k+5} - x_{k+2}) \times (x_{k+5} - x_{k+1})(x_{k+5} - x_k);$$

$$\omega_{k+6}^R(x) = \frac{c_{k+6}(x)}{d_{k+6}},$$

$$c_{k+6}(x) = (x - x_{k+5})(x - x_{k+4})(x - x_{k+3}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+6} = (x_{k+6} - x_{k+5})(x_{k+6} - x_{k+4}) \times (x_{k+6} - x_{k+3})(x_{k+6} - x_{k+2}) \times (x_{k+6} - x_{k+1})(x_{k+6} - x_k).$$

Approximations with these basis splines can be applied on the next grid intervals  $[x_k, x_{k+1}]$ ,  $k = 0, \dots, n - 6$ .

Consider the approximation with the right-left basis splines. Let  $r_1 = 2$ ,  $r = 5$ , in this case, on the interval  $[x_k, x_{k+1}]$  formula (1) takes the form:

$$\tilde{u}^{RL}(x) = \sum_{j=k-1}^{k+5} u(x_j) \omega_j^{RL}(x), \quad x \in [x_k, x_{k+1}],$$

where the right-left basis splines  $\omega_j^{RL}(x)$  have the form:

$$\omega_{k+5}^{RL}(x) = \frac{c_{k+5}(x)}{d_{k+5}},$$

$$c_{k+5}(x) = (x - x_{k+4})(x - x_{k+3})(x - x_{k+2}) \times (x - x_{k+1})(x - x_k)(x - x_{k-1}),$$

$$d_{k+5} = (x_{k+5} - x_{k+4})(x_{k+5} - x_{k+3}) \times (x_{k+5} - x_{k+2})(x_{k+5} - x_{k+1}) \times (x_{k+5} - x_k)(x_{k+5} - x_{k-1});$$

$$\omega_{k+4}^{RL}(x) = \frac{c_{k+4}(x)}{d_{k+4}},$$

$$c_{k+4}(x) = (x - x_{k+5})(x - x_{k+3})(x - x_{k+2}) \times (x - x_{k+1})(x - x_k)(x - x_{k-1}),$$

$$d_{k+4} = (x_{k+4} - x_{k+5})(x_{k+4} - x_{k+3}) \times (x_{k+4} - x_{k+2})(x_{k+4} - x_{k+1}) \times (x_{k+4} - x_k)(x_{k+4} - x_{k-1});$$

$$\omega_{k+3}^{RL}(x) = \frac{c_{k+3}(x)}{d_{k+3}},$$

$$c_{k+3}(x) = (x - x_{k+5})(x - x_{k+4})(x - x_{k+2}) \times (x - x_{k+1})(x - x_k)(x - x_{k-1}),$$

$$d_{k+3} = (x_{k+3} - x_{k+5})(x_{k+3} - x_{k+4}) \times (x_{k+3} - x_{k+2})(x_{k+3} - x_{k+1}) \times (x_{k+3} - x_k)(x_{k+3} - x_{k-1});$$

$$\omega_{k+2}^{RL}(x) = \frac{c_{k+2}(x)}{d_{k+2}},$$

$$\begin{aligned} c_{k+2}(x) &= (x - x_{k+5})(x - x_{k+4})(x - x_{k+3}) \\ &\times (x - x_{k+1})(x - x_k)(x - x_{k-1}), \\ d_{k+2} &= (x_{k+2} - x_{k+5})(x_{k+2} - x_{k+4}) \\ &\times (x_{k+2} - x_{k+3})(x_{k+2} - x_{k+1}) \\ &\times (x_{k+2} - x_k)(x_{k+2} - x_{k-1}); \end{aligned}$$

$$\omega_{k+1}^{RL}(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$\begin{aligned} c_{k+1}(x) &= (x - x_{k+5})(x - x_{k+4})(x - x_{k+3}) \\ &\times (x - x_{k+2})(x - x_k)(x - x_{k-1}), \\ d_{k+1} &= (x_{k+1} - x_{k+5})(x_{k+1} - x_{k+4}) \\ &\times (x_{k+1} - x_{k+3})(x_{k+1} - x_{k+2}) \\ &\times (x_{k+1} - x_k)(x_{k+1} - x_{k-1}); \end{aligned}$$

$$\omega_k^{RL}(x) = \frac{c_k(x)}{d_k},$$

$$\begin{aligned} c_k(x) &= (x - x_{k+5})(x - x_{k+4})(x - x_{k+3}) \\ &\times (x - x_{k+2})(x - x_{k+1})(x - x_{k-1}), \\ d_k &= (x_k - x_{k+5})(x_k - x_{k+4}) \\ &\times (x_k - x_{k+3})(x_k - x_{k+2}) \\ &\times (x_k - x_{k+1})(x_k - x_{k-1}); \end{aligned}$$

$$\omega_{k-1}^{RL}(x) = \frac{c_{k-1}(x)}{d_{k-1}},$$

$$\begin{aligned} c_{k-1}(x) &= (x - x_{k+5})(x - x_{k+4})(x - x_{k+3}) \\ &\times (x - x_{k+2})(x - x_{k+1})(x - x_k), \\ d_{k-1} &= (x_{k-1} - x_{k+5})(x_{k-1} - x_{k+4}) \\ &\times (x_{k-1} - x_{k+3})(x_{k-1} - x_{k+2}) \\ &\times (x_{k-1} - x_{k+1})(x_{k-1} - x_k). \end{aligned}$$

Approximations with these basis splines can be applied on the next grid intervals  $[x_k, x_{k+1}]$ ,  $k = 1, \dots, n - 5$ .

Consider the approximation with the right-left-left basis splines. Let  $r_1 = 3$ ,  $r = 4$ . In this case, on the interval  $[x_k, x_{k+1}]$  formula (1) takes the form:

$$\begin{aligned} \tilde{u}^{RLL}(x) &= \sum_{j=k-2}^{k+4} u(x_j)\omega_j^{RLL}(x), \\ x &\in [x_k, x_{k+1}], \end{aligned}$$

where the right-left-left basis splines  $\omega_j^{RLL}(x)$  have the form:

$$\omega_{k+4}^{RLL}(x) = \frac{c_{k+4}(x)}{d_{k+4}},$$

$$\begin{aligned} c_{k+4}(x) &= (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}) \\ &\times (x - x_k)(x - x_{k-1})(x - x_{k-2}), \\ d_{k+4} &= (x_{k+4} - x_{k+3})(x_{k+4} - x_{k+2}) \end{aligned}$$

$$\begin{aligned} &\times (x_{k+4} - x_{k+1})(x_{k+4} - x_k) \\ &\times (x_{k+4} - x_{k-1})(x_{k+4} - x_{k-2}); \end{aligned}$$

$$\omega_{k+3}^{RLL}(x) = \frac{c_{k+3}(x)}{d_{k+3}},$$

$$\begin{aligned} c_{k+3}(x) &= (x - x_{k+4})(x - x_{k+2})(x - x_{k+1}) \\ &\times (x - x_k)(x - x_{k-1})(x - x_{k-2}), \\ d_{k+3} &= (x_{k+3} - x_{k+4})(x_{k+3} - x_{k+2}) \\ &\times (x_{k+3} - x_{k+1})(x_{k+3} - x_k) \\ &\times (x_{k+3} - x_{k-1})(x_{k+3} - x_{k-2}); \end{aligned}$$

$$\omega_{k+2}^{RLL}(x) = \frac{c_{k+2}(x)}{d_{k+2}},$$

$$\begin{aligned} c_{k+2}(x) &= (x - x_{k+4})(x - x_{k+3})(x - x_{k+1}) \\ &\times (x - x_k)(x - x_{k-1})(x - x_{k-2}), \\ d_{k+2} &= (x_{k+2} - x_{k+4})(x_{k+2} - x_{k+3}) \\ &\times (x_{k+2} - x_{k+1})(x_{k+2} - x_k) \\ &\times (x_{k+2} - x_{k-1})(x_{k+2} - x_{k-2}); \end{aligned}$$

$$\omega_{k+1}^{RLL}(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$\begin{aligned} c_{k+1}(x) &= (x - x_{k+4})(x - x_{k+3})(x - x_{k+2}) \\ &\times (x - x_k)(x - x_{k-1})(x - x_{k-2}), \\ d_{k+1} &= (x_{k+1} - x_{k+4})(x_{k+1} - x_{k+3}) \\ &\times (x_{k+1} - x_{k+2})(x_{k+1} - x_k) \\ &\times (x_{k+1} - x_{k-1})(x_{k+1} - x_{k-2}); \end{aligned}$$

$$\omega_k^{RLL}(x) = \frac{c_k(x)}{d_k},$$

$$\begin{aligned} c_k(x) &= (x - x_{k+4})(x - x_{k+3})(x - x_{k+2}) \\ &\times (x - x_{k+1})(x - x_{k-1})(x - x_{k-2}), \\ d_k &= (x_k - x_{k+4})(x_k - x_{k+3}) \\ &\times (x_k - x_{k+2})(x_k - x_{k+1}) \\ &\times (x_k - x_{k-1})(x_k - x_{k-2}); \end{aligned}$$

$$\omega_{k-1}^{RLL}(x) = \frac{c_{k-1}(x)}{d_{k-1}},$$

$$\begin{aligned} c_{k-1}(x) &= (x - x_{k+4})(x - x_{k+3})(x - x_{k+2}) \\ &\times (x - x_{k-1})(x - x_k)(x - x_{k-2}), \\ d_{k-1} &= (x_{k-1} - x_{k+4})(x_{k-1} - x_{k+3}) \\ &\times (x_{k-1} - x_{k+2})(x_{k-1} - x_{k+1}) \\ &\times (x_{k-1} - x_k)(x_{k-1} - x_{k-2}); \end{aligned}$$

$$\omega_{k-2}^{RLL}(x) = \frac{c_{k-2}(x)}{d_{k-2}},$$

$$\begin{aligned} c_{k-2}(x) &= (x - x_{k+4})(x - x_{k+3})(x - x_{k+2}) \\ &\times (x - x_{k+1})(x - x_k)(x - x_{k-1}), \\ d_{k-2} &= (x_{k-2} - x_{k+4})(x_{k-2} - x_{k+3}) \\ &\times (x_{k-2} - x_{k+2})(x_{k-2} - x_{k+1}) \\ &\times (x_{k-2} - x_k)(x_{k-2} - x_{k-1}). \end{aligned}$$



Approximations with these basis splines can be applied on the next grid intervals  $[x_k, x_{k+1}]$ ,  $k = 2, \dots, n - 4$ .

When approximating a function with the splines of the 7<sup>th</sup> order of approximation, the next Theorem is valid.

*Theorem.* If  $\text{supp } \omega_k = [x_{k-1}, x_{k+6}]$ , then the following inequality is valid:

$$|u(x) - \tilde{u}^L(x)|_{x \in [x_k, x_{k+1}]} \leq h^7 \frac{95.842}{7!} \|u^{(7)}\|_{C[x_{k-5}, x_{k+1}]}.$$

If  $\text{supp } \omega_k = [x_{k-2}, x_{k+5}]$ , then the following inequality is valid:

$$|u(x) - \tilde{u}^{LR}(x)|_{x \in [x_k, x_{k+1}]} \leq h^7 \frac{23.149}{7!} \|u^{(7)}\|_{C[x_{k-4}, x_{k+2}]}.$$

If  $\text{supp } \omega_k = [x_{k-3}, x_{k+4}]$ , then the following approximation estimate is valid:

$$|u(x) - \tilde{u}^M(x)|_{x \in [x_k, x_{k+1}]} \leq h^7 \frac{12.359}{7!} \|u^{(7)}\|_{C[x_{k-3}, x_{k+3}]}.$$

If  $\text{supp } \omega_k = [x_{k-4}, x_{k+3}]$ , then the following approximation estimate is valid:

$$|u(x) - \tilde{u}^{RLL}(x)|_{x \in [x_k, x_{k+1}]} \leq h^7 \frac{12.359}{7!} \|u^{(7)}\|_{C[x_{k-2}, x_{k+4}]}.$$

If  $\text{supp } \omega_k = [x_{k-5}, x_{k+2}]$ , then the following approximation estimate is valid:

$$|u(x) - \tilde{u}^{RL}(x)|_{x \in [x_k, x_{k+1}]} \leq h^7 \frac{23.149}{7!} \|u^{(7)}\|_{C[x_{k-1}, x_{k+5}]}.$$

If  $\text{supp } \omega_k = [x_{k-6}, x_{k+1}]$ , then the following approximation estimate is valid:

$$|u(x) - \tilde{u}^R(x)|_{x \in [x_k, x_{k+1}]} \leq h^7 \frac{95.842}{7!} \|u^{(7)}\|_{C[x_k, x_{k+6}]}.$$

*Proof.* In the case of approximating the function  $u$  on the interval  $[x_k, x_{k+1}]$  near the left end of the interval  $[a, b]$ , we use the right basis splines

$$\tilde{u}(x) = \sum_{j=k}^{k+6} u(x_j) \omega_j^R(x) dx, x \in [x_k, x_{k+1}].$$

First, we estimate the approximation error on the interval  $[x_k, x_{k+1}]$  when the right basis splines are used. Using the formula of the remainder term of the interpolation polynomial that solves the Lagrange interpolation problem, we obtain the relation

$$u(x) - \tilde{u}(x) = \frac{1}{7!} (x - x_k) \dots (x - x_{k+6}) u^{(7)}(\xi), \quad \xi \in [x_{k-5}, x_{k+1}].$$

There is a product  $(x - x_k) \dots (x - x_{k+6})$  in the error estimate. Let the ordered grid of nodes  $\{x_k\}$  be uniform with step  $h$ . Let us estimate the product of factors  $(x - x_k) \dots (x - x_{k+6})$ .

Thus, estimating the maximum of the expression  $\frac{1}{7!} (x - x_k) \dots (x - x_{k+6}) u^{(7)}(\xi)$ , where  $\xi \in [x_k, x_{k+6}]$ , we obtain

$$\|u(x) - \tilde{u}(x)\|_{C[x_k, x_{k+1}]} \leq K h^7 \|u^{(7)}\|_{C[x_k, x_{k+6}]}.$$

Similarly, we obtain an approximation estimate on the grid interval  $[x_k, x_{k+1}]$  with the left and middle splines.

This completes the proof of the theorem.

*Remark 1.* The approximation on the interval.

If the interval  $[a, b]$  is divided into 6 grid intervals, then it is possible to construct an approximation on the entire interval using the previously presented approximations on the intervals  $[x_k, x_{k+1}]$ , as follows:

- for  $k = 0$  we apply the approximation  $\tilde{u}^R(x)$ ;
- for  $k = 1$  we apply the approximation  $\tilde{u}^{RL}(x)$ ;
- for  $k = 2$  we apply the approximation  $\tilde{u}^{RLL}(x)$ ;
- for  $k = 3$  we apply the approximation  $\tilde{u}^M(x)$ ;
- for  $k = 4$  we apply the approximation  $\tilde{u}^{LR}(x)$ ;
- for  $k = 5$  we apply the approximation  $\tilde{u}^L(x)$ .

*Remark 2.* The stability.

Let  $a = 0$ ,  $b = 1$ , and  $x_{k+1} - x_k = h = \text{const}$ . Suppose  $|K(x, s)| < \rho < 1$ , when  $0 \leq x \leq 1$ ,  $0 \leq s \leq 1$ . The constant in the stability inequality in the case of splines of the seventh order of approximation is calculated in the same way as it was done for the splines of the second order of approximation. To calculate the constant in the stability inequality, we need the following values:

$$\sum_{j=k-2}^{k+4} \int_{x_k}^{x_{k+1}} |\omega_j^{RLL}(x)| dx \approx 1.321 h,$$

$$\begin{aligned} \sum_{j=k-1}^{k+5} \int_{x_k}^{x_{k+1}} |\omega_j^{RL}(x)| dx &\approx 1.635 h, \\ \sum_{j=k}^{k+6} \int_{x_k}^{x_{k+1}} |\omega_j^R(x)| dx &\approx 3.233 h, \\ \sum_{j=k-3}^{k+3} \int_{x_k}^{x_{k+1}} |\omega_j^M(x)| dx &\approx 1.321 h, \\ \sum_{j=k-4}^{k+2} \int_{x_k}^{x_{k+1}} |\omega_j^{LR}(x)| dx &\approx 1.635h, \\ \sum_{j=k-5}^{k+1} \int_{x_k}^{x_{k+1}} |\omega_j^L(x)| dx &\approx 3.233 h. \end{aligned}$$

Taking into account the inequalities given above, we obtain the constants

$$s_1 = (2 \cdot 3.233h + (n - 2) \cdot 1.321h)/n,$$

$$s_2 = (3.233h + 1.635h + 1.321h) \cdot 2 + (n - 6) \cdot 1.321/n.$$

Thus, we have the estimations

$$|u(x_j)| \leq \frac{1}{1 - \rho s_i} |f(x_j)|, \quad j = 0, \dots, n, i = 1, 2.$$

Suppose that  $|\rho s_i| < 1$ . In this case, taking into account the approximation theorem and the inequalities given above, we can see that the approximate solution obtained with the splines of the seventh order of approximation tends to the solution to the Fredholm equation.

### 3 Problem Solution

In this section, we discuss the solution of the integral equation of the second kind

$$Au \equiv u(x) - \int_a^b K(x, s)u(s)ds = f(x), \quad x \in [a, b],$$

with the local splines of the seventh order of approximation.

Let us choose an integer  $n \geq 7$ . We build a grid of nodes  $\{s_j\}$ .

The function  $g(s) = K(x, s)u(s), s \in [s_k, s_{k+1}]$ , can be approximated with the expression:  $g(s) \approx \tilde{g}(s) = K(x, s)\tilde{u}(s)$ . Let us denote  $c_j = u(s_j)$ .

We represent the integral in the form

$$\int_a^b g(s)ds = \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} g(s)ds.$$

Using the results from the second section, we can reduce the integral equation to the solution of a system of linear algebraic equations. To do this, we put  $x = x_m, m = 0, \dots, n - 1$ , in the equation

$$\begin{aligned} u(x) - \sum_{k=0}^2 \sum_{j=k}^{k+6} c_j \int_{s_k}^{s_{k+1}} K(x, s)\omega_j^R(s)ds \\ - \sum_{k=3}^{n-3} \sum_{j=k-3}^{k+3} c_j \int_{s_k}^{s_{k+1}} K(x, s)\omega_j^M(s)ds \\ - \sum_{k=n-2}^{n-1} \sum_{j=n-3}^{n-1} c_j \int_{s_k}^{s_{k+1}} K(x, s)\omega_j^L(s)ds = f(x). \end{aligned}$$

And now we have to solve the system of linear algebraic equations

$$\begin{aligned} u(x_m) - \sum_{k=0}^2 \sum_{j=k}^{k+6} c_j \int_{s_k}^{s_{k+1}} K(x_m, s)\omega_j^R(s)ds + \\ \sum_{k=3}^{n-2} \sum_{j=k-3}^{k+3} c_j \int_{s_k}^{s_{k+1}} K(x_m, s)\omega_j^M(s)ds + \\ \sum_{k=n-2}^{n-1} \sum_{j=n-3}^{n-1} c_j \int_{s_k}^{s_{k+1}} K(x_m, s)\omega_j^L(s)ds = f(x_m), \\ m = 0, \dots, n. \end{aligned}$$

We can also apply a more detailed approximation (as shown in sections 2.2.1-2.2.6) on the grid intervals adjacent to the boundaries of the interval  $[a, b]$ . We assume that the integral  $\int_{s_k}^{s_{k+1}} v(s)ds = \int_{s_k}^{s_{k+1}} K(x, s)\omega_j(s)ds$  can be computed exactly. Otherwise, we can use quadrature formulas. In this case, it is necessary to take into account the error of the applied quadrature formula. We can use, for example, the Simpson's compound formula:

$$\begin{aligned} \int_{s_k}^{s_{k+1}} v(x)dx = \\ = \frac{(s_{k+1} - s_k)}{3N} (v_0 + 4(v_1 + v_3 + \dots + v_{N-1}) \\ + 2(v_2 + v_4 + \dots + v_{N-2}) + v_N) + R_N(v). \end{aligned}$$

Here  $N$  is even,  $N \geq 2$ .

The formula for the remainder of the Simpson's compound quadrature rule is well known. In the case of the interval  $[s_k, s_{k+1}]$ , it has the form

$$R_n(v) = \frac{-(s_{k+1} - s_k)^5}{180N^4} v^{(IV)}(\xi),$$

$$\xi \in [s_k, s_{k+1}].$$

Proceeding from this formula and taking into account the approximation theorem for the splines of the seventh order, it is easy to get a priori estimation of the number of nodes of the Simpson's quadrature formula.

Let us calculate the integral from the Runge function  $\int_0^{0.4} v(x)dx$ , where  $v(x) = \frac{1}{1+25x^2}$ .

We can obtain

$$\max_{x \in [0, 0.4]} |v^{(IV)}(x)| \leq 15000,$$

$$\max_{x \in [0, 0.4]} |v^{(VII)}(x)| < 0.344 \cdot 10^9.$$

We can easily calculate  $N = 2$ , and  $|R_2(v)| \leq 0.46 \cdot 10^{-6}$ . Note that this number is less than the theoretical error of integration (which follows from the Theorem). We can easily obtain that the theoretical error of the integration is about  $0.43 \cdot 10^{-4}$ .

After the approximate solution at the grid nodes is obtained, we can use the expressions

$$u(x) = \sum_{k=0}^2 \sum_{j=k}^{k+6} c_j \int_{s_k}^{s_{k+1}} K(x, s) \omega_j^R(s) ds$$

$$+ \sum_{k=3}^{n-3} \sum_{j=k-3}^{k+3} c_j \int_{s_k}^{s_{k+1}} K(x, s) \omega_j^M(s) ds$$

$$+ \sum_{k=n-2}^{n-1} \sum_{j=n-3}^{n-1} c_j \int_{s_k}^{s_{k+1}} K(x, s) \omega_j^L(s) ds + f(x).$$

to solve the problems that arise further, for example, the construction of a plot of the solution.

Let us consider two examples of the application of splines of the seventh order of approximation and splines of the second order of approximation to solve the Fredholm integral equation of the second kind.

*Example 1.* Consider the equation

$$u(x) - 0.1 \int_0^1 \sin\left(\frac{xs}{5}\right) u(s) ds = x \exp(x),$$

$$x \in [0, 1].$$

First, let us construct an ordered grid of nodes  $x_j$  with step  $h = 1/n$  ( $n = 10$ ) on the interval  $[a, b]$ . Using splines of the seventh order of approximation, we calculate the approximate values of the solution at these nodes. Next, we construct a sequence of refining grids on the interval  $[a, b]$  as follows: we divide each grid interval in half. The division points are added to the nodes of the previous grid.

Thus, we get a new grid of nodes. We compare the values of the approximate solution at some grid nodes.

Table 1 presents the approximate values of the solution at some grid nodes when using splines of the seventh order of approximation when  $n = 10, 20$ . Table 2 presents the approximate values of the solution at the same grid nodes when using splines of the second order of approximation when  $n = 10, 20, 50$ .

Table 1. The Approximate Values of the Solution when Splines of the Seventh Order of Approximation were used

$x_j$	Splines of the Seventh Order of Approximation	
	$n = 10$	$n = 20$
0	0	0
0.2	0.24717242	0.24717241
0.4	0.60251062	0.60251061
0.6	1.10193492	1.10193491
0.8	1.79197032	1.79197030
1.0	2.73268142	2.73268140

Table 2. The Approximate Values of the Solution when Splines of the Second Order of Approximation were used

$x_j$	Splines of the Second Order of Approximation		
	$n = 10$	$n = 20$	$n = 50$
0	0	0	0
0.2	0.24717235	0.24717240	0.24717241
0.4	0.60251051	0.60251059	0.60251061
0.6	1.10193482	1.10193489	1.10193490
0.8	1.79197029	1.79197031	1.79197030
1.0	2.73268156	2.73268145	2.73268140

Having solved the system of linear algebraic equations, we obtain the framework of an approximate solution. The obtained values of the approximate solution at the grid nodes are marked with circles. Further, using the integral equation and

the found approximate values of the function, we connect the found values with a line. The Maple package was used in the calculations. The plot of the approximate solution with the splines of the second order of approximation when  $n = 10$  is given in Figure 1. The nodes are marked along the abscissa axis.

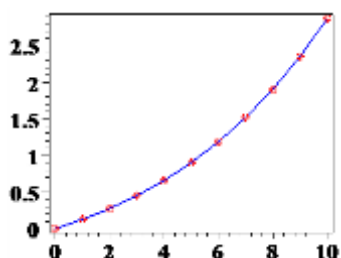


Fig. 1: The plot of the approximate solution when  $n = 10$ .

Example 2. Consider the equation

$$u(x) - 0.7 \int_0^1 \sin(x + s/5) u(s) ds = \sin(x).$$

As is known, in order to obtain an approximate solution of the integral equation, it is necessary to apply different methods and calculate with a different number of grid nodes. Let us start with calculations using splines of the second order of approximation. Table 3 presents the approximate values of the solution at grid nodes when using splines of the second order of approximation when  $n = 10, 20$ . Table 4 presents the approximate values of the solution at the same grid nodes when using splines of the second order of approximation when  $n = 40, 80$ . Table 5 presents the approximate values of the solution at the grid nodes when splines of the second order of approximation when  $n = 640$  and splines of the seventh order of approximation when  $n = 640$  were used. Table 6 shows the approximate values of the solution when splines of the seventh order of approximation were used ( $n = 20, n = 40$ ).

Table 3. The Approximate Values of the Solution when Splines of the Second Order of Approximation were used

$x_j$	Splines of the Second Order of Approximation	
	$n = 10$	$n = 20$
0	0.06744487	0.06763105
0.2	0.36908780	0.36939306
0.4	0.65601636	0.65642854
0.6	0.91679162	0.91729428
0.8	1.14101729	1.14159039
1.0	1.31975420	1.32037489

Table 4. The Approximate Values of the Solution when Splines of the Second Order of Approximation were used

$x_j$	Splines of the Second Order of Approximation	
	$n = 40$	$n = 80$
0	0.06767764	0.06768929
0.2	0.36946943	0.36948853
0.4	0.65653166	0.65655744
0.6	0.91742003	0.91745147
0.8	1.14173376	1.14176961
1.0	1.32053017	1.32056899

Table 5. The Approximate Values of the Solution when Splines of the second order of approximation and the seventh of Approximation were used

$x_j$	Splines	
	Splines of the Second Order of Approximation $n = 640$	Splines of the Seventh Order of Approximation $n = 10$
0	0.067693111	0.06769317
0.2	0.36949480	0.36949490
0.4	0.65656590	0.65656604
0.6	0.91746179	0.91746195
0.8	1.14178137	1.14178156
1.0	1.32058173	1.32058194

Table 6. The Approximate Values of the Solution when Splines of the Seventh Order of Approximation were used

$x_j$	Splines of the Seventh Order of Approximation	
	$n = 20$	$n = 40$
0	0.06769317	0.067689317
0.2	0.36949490	0.36949490
0.4	0.65656603	0.65656604
0.6	0.91746195	0.91746195
0.8	1.14178156	1.14178156
1.0	1.32058194	1.32058194

The absolute values of the differences between the solution found on the grid of nodes ( $n = 10$ ) and the solution found on the grid of nodes ( $n = 20$ ) are shown in Figure 2 (blue line); absolute values of the differences between the solution found on the grid ( $n = 20$ ) and the solution found on the grid ( $n = 40$ ) are shown in Figure 2 (green line).

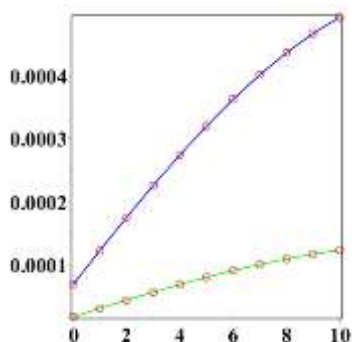


Fig. 2: The absolute values of the differences between the solution found on the grid of nodes ( $n = 10$ ) and the solution found on the grid of nodes ( $n = 20$ ) (blue line); absolute values of the differences between the solution found on the grid ( $n = 20$ ) and the solution found on the grid ( $n = 40$ ) (green line).

Figure 3 shows the absolute values of the differences of the solution found on the grid ( $n = 10$ ) and the solution found on the grid ( $n = 20$ ) when the splines of the seventh order of approximation were used.

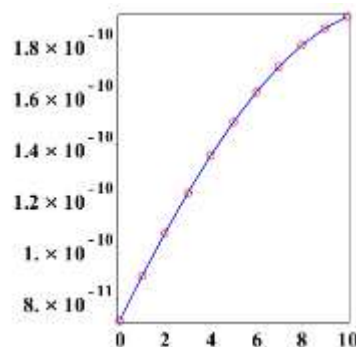


Fig. 3: The absolute values of the differences of the solution found on the grid ( $n = 10$ ) and the solution found on the grid ( $n = 20$ ) when the splines of the seventh order of approximation were used.

Figure 4 shows the absolute values of the differences of the solution found on the grid ( $n = 20$ ) and the solution found on the grid ( $n = 40$ ) when the splines of the seventh order of approximation were used.

Figure 5 shows the plot of the approximate solution when the splines of the seventh order of approximation were used and  $n = 40$ .

The calculation results show that splines of the seventh order of approximation give an approximate solution at the grid nodes with eight correct digits in the mantissa when  $n = 20$ .

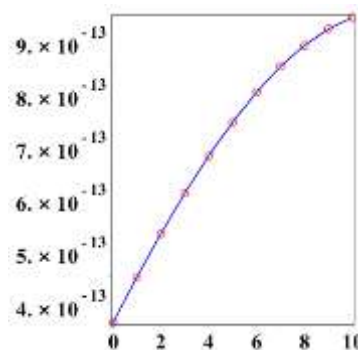


Fig. 4: The absolute values of the differences of the solution found on the grid ( $n = 20$ ) and the solution found on the grid ( $n = 40$ ) when the splines of the seventh order of approximation were used.

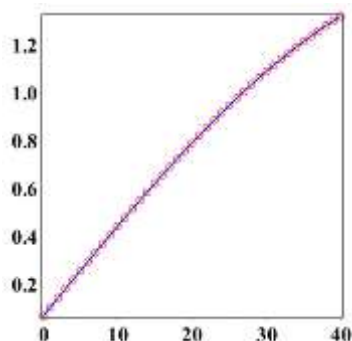


Fig. 5: The plot of the approximate solution when the splines of the seventh order of approximation were used ( $n = 40$ ).

Comparing the results in the second and third columns of Table 5, we see that the spline of the second order of approximation provides five correct digits in the mantissa when we use 640 grid nodes. At the same time, to achieve the same accuracy using splines of the seventh order of approximation, it is enough to use 10 grid nodes.

#### 4 Conclusion

In this paper, we consider the stability of the solution of the integral equations of the second kind using splines of the second and the seventh order of approximation. It should be noted that with the same number of grid nodes, polynomial splines of the seventh order of approximation provide a smaller error compared to splines of the second order of approximation. When using splines of the seventh order of approximation, a large number of nodes is not recommended.

In the future, numerical schemes for integro-differential equations will be constructed.

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#### Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

-I. G. Burova developed the theoretical part,  
 -G. O. Alcybeev executed the numerical experiments,  
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