On an Attraction Basin of the Generalized Kapitsa's Problem

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Abstract The problem of the generalized Kapitsa pendulum on the stability of the vertical position of the rod under the vertical vibration of the support was studied in various settings. A vertical deformable rod with a free upper end and clamped or simply supported lower end under the action of harmonic or stationary random vibrations of the support is considered. We model the rod as a system with several degree of freedom. The conditions for stability of the upper vertical position of the pendulum are found. Both bending and longitudinal vibrations of the bar are taken into account. We found the attraction basin of the stable vertical position.

Key words: Kapitsa's pendulum, stability, attraction basin, two-scale asymptotic expansion, harmonic and random vibrations, flexible rod, single mode approximation

1 Introduction

Interest in the problem of pendulum oscillations was born 300 years ago in the works of Galileo, who studied the periods of pendulum oscillations. It was A. Stephenson [12] who in 1908 first drew attention to one of the very interesting types of pendulum oscillations, namely, the stability of a pendulum in a gravity field in the upward position with vertical support vibrations.

With the development of high-energy physics, the problems involving oscillatory behaviour of objects with different time scales have received practical application and have attracted a vivid attention of researchers. In 1951 P.L. Kapitsa [7, 8] carried out

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various statements of theoretical and experimental studies of the oscillations of an inverted pendulum. It is known that the problems of oscillations of a pendulum with a vibrating support lead to the Mathieu equation which can be solved only in terms of elliptic functions. Kapitsa made an additional assumption of small oscillation amplitude of the support and considered a type of motion in which the period of the support oscillation is much less than the oscillation period of the pendulum itself. Under these assumptions, the pendulum can stand, without falling, in the upward position which was confirmed by a number of experiments described in the Kapitsa work. In the Kapitsa works one find the theory of calculation of the pendulum oscillation period, the restoring moment acting on the pendulum deviated from the upper equilibrium position by a finite angle, as well as the equilibrium condition itself and an accuracy estimate under the assumption of small oscillation amplitude of the pivot point. The equilibrium condition occurs with sufficiently intense vibrations of the support.

The problem of the Kapitsa pendulum as well as similar beautiful and instructive phenomena of dynamic stability and instability associated with vibrations were included in the monographs by I.I. Blekhman [4, 5].

The present paper suggests the boundaries of the attraction basin of the upper stable position of the pendulum found by the method of two-scale asymptotic expansion. The solutions of the generalized problem of oscillations of the Kapitsa pendulum are investigated as they are important for practical application, too. Even P.L. Kapitsa in his work drew attention to the parameters of a pendulum suitable for practical experiments and predicted that bending vibrations at resonance frequencies can grow for a thin rod.

This paper gives attention to the generalized formulations of the problem in which the pendulum rod is not an absolutely rigid body. The flexible pendulum is assumed to be a homogeneous rod that obeys the hypotheses adopted for the Bernoulli - Euler beam. To analyze the solution of the problem, a series expansion in eigenforms of an auxiliary boundary value problem associated with free transverse vibrations of a rod compressed by a longitudinal force is used. Here the stability condition is found from the system of equations obtained by the method of two-scale asymptotic expansions.

A pendulum in the form of a vertical elastic rod is considered, which can be unstable not only in the case of a hinged support of the lower end, but also in the case of rigid fixation (provided that the rod is long enough). The influence of the propagation of longitudinal waves along the rod is investigated and the attraction basins are found [3].

The generalised Kapitsa problem is also considered in the case when the vibrations of the vertical support is a stationary random process [16]. As in the case of high vibration levels, the vertical position of the rod is stable and we determine the corresponding attraction basin.

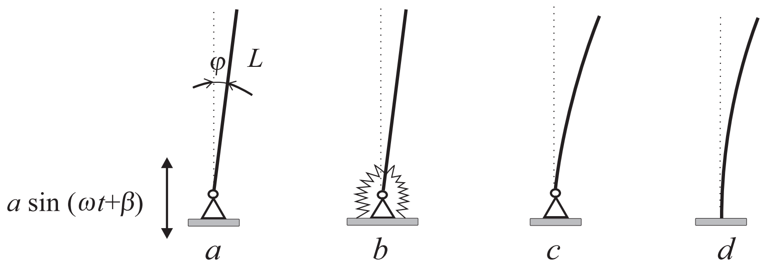


Fig. 1 The Kapitsa's pendulum and its generalized models: a. Classical model; b. Flexible support; c. Flexible rod with a hinged lower end; d. Flexible rod with a clamped lower end

2 Classical Model of the Kapitsa's Pendulum

2.1 Stability of the Kapitsa's Pendulum

We consider a pendulum as a thin homogeneous rigid rod of length L, see Fig. 1a. Its motion in the movable co-ordinate system is described by the equation

$$J\frac{d^2\varphi}{d\tilde{t}^2} + n_1\frac{d\varphi}{d\tilde{t}} - m\frac{L}{2}(g - a\omega^2\sin(\omega\tilde{t}+\beta))\sin\varphi = 0, \tag{1}$$

where $\varphi(\tilde{t})$ is the angle between a rod and a vertical axis; $n_1,\ J = mL^2/3,\ m,\ g$ are the damping coefficient, the inertia moment of a rod, its mass, and the gravitational acceleration; respectively, a, ω, β are the amplitude, frequency and initial phase of the support vibration.

The limitation of small amplitude $a \ll L$ of the support vibration is introduced. Additionally, it is known [7] that for the fixed values of L and a the stability condition for the Kapitsa effect is fulfilled for the sufficiently high frequency $\omega \gg 1$. For this reason, for the following analysis is convenient to write down Eq.(1) to the dimensionless form:

$$\ddot{\varphi} + \varepsilon n\dot{\varphi} - (\varepsilon^2 q - \varepsilon\sin(t+\beta))\sin\varphi = 0, \tag{2}$$

where

$$t = \omega\tilde{t}, \quad n = \frac{2n_1}{mLa\omega}, \quad q = \frac{2Lg}{3a^2\omega^2}, \quad \varepsilon = \frac{3\delta}{2}, \quad \delta = \frac{a}{L}. \tag{3}$$

Here q is the loading parameter, ε is the small parameter. A derivative with respect to time t is denoted by a dot. We introduce the relative acceleration of the support vibration κ as the critical parameter which ensures the pendulum stability

$$\kappa = \frac{a\omega^2}{g} = \frac{2}{3\delta q}. \tag{4}$$

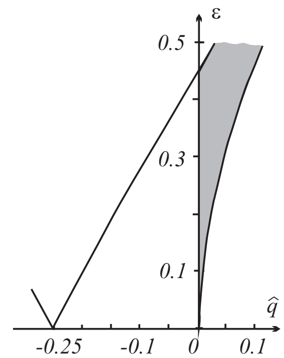


Fig. 2 Part of the Anis-Strett diagram for the equation $\ddot{\varphi} - (\hat{q} - \varepsilon \sin t)\varphi = 0$

For small φ (namely, for $\sin\varphi \approx \varphi$) Eq.(2) is the Mathieu equation and for small $q > 0$ the solution $\varphi \equiv 0$ is stable under the condition [1] (see Fig. 2)

$$q < 1/2, \quad \kappa > \frac{4}{3\delta}. \tag{5}$$

2.2 Attraction Basin of the Solution of the Kapitsa's Pendulum

Now we proceed to the attraction basin of this solution and consider the Cauchy problem consisting of Eq.(2) and the initial conditions

$$\varphi(0) = \varphi_0, \qquad \dot{\varphi}(0) = 0. \tag{6}$$

We seek an asymptotic solution of Eq.(2) as a two-scale expansion [6]:

$$\varphi(t,\theta,\varepsilon) = \sum_{m=0}^{\infty}(U_m(\theta) + V_m(t,\theta))\varepsilon^m, \quad \int_0^{2\pi} V_m(t,\theta))dt = 0, \quad m = 0, 1, \ldots, \tag{7}$$

where $\theta = \varepsilon t$ is the slow time and

$$\dot{\varphi} = \frac{\partial\varphi}{\partial t} + \varepsilon\frac{\partial\varphi}{\partial\theta}, \quad \ddot{\varphi} = \frac{\partial^2\varphi}{\partial t^2} + 2\varepsilon\frac{\partial^2\varphi}{\partial t\partial\theta} + \varepsilon^2\frac{\partial^2\varphi}{\partial\theta^2}. \tag{8}$$

An expansion of Eq.(2) in powers of ε yields consecutively

$$V_0(t,\theta) = 0, \quad V_1(t,\theta) = \sin U_0 \sin(t+\beta), \quad \frac{\partial^2 V_2}{\partial t^2} + H(\theta,t) = 0, \tag{9}$$

with

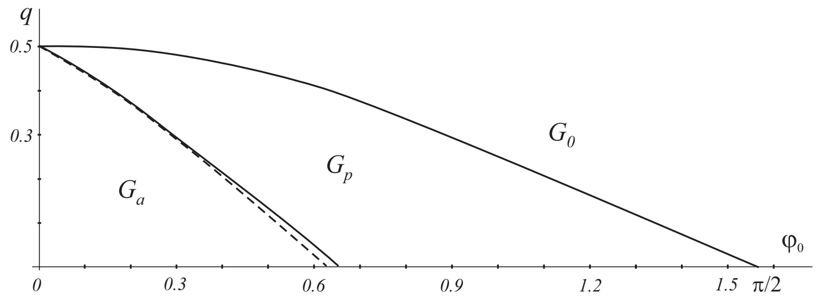


Fig. 3 The attraction basins

$$H = 2\frac{\partial^2 V_1}{\partial t \partial \theta} + \frac{d^2 U_0}{d\theta^2} + n\frac{dU_0}{d\theta} - q \sin U_0 + (U_1 + V_1)\cos U_0 \sin(t+\beta). \tag{10}$$

According to (7) the average value in t of function $H(t,\theta)$ is to be equal zero that gives an equation for function $U_0(\theta)$

$$\frac{d^2 U_0}{d\theta^2} + n\frac{dU_0}{d\theta} + F(U_0) = 0, \quad F(U_0) = ((1/2)\cos U_0 - q)\sin U_0. \tag{11}$$

Due to relation $\dot{\varphi} = \varepsilon(dU_0/d\theta + \partial V_1/\partial t) + O(\varepsilon^2) = 0$, we solve Eq.(11) with initial conditions

$$U_0 = \varphi_0, \quad U_0' = dU_0/d\theta = -\sin\varphi_0 \cos\beta \quad \text{for} \quad \theta = 0. \tag{12}$$

The problem (11), (12) is the zero asymptotic approximation of the exact problem (2), (6).

For a definiteness we take $\alpha = 0.01$, $n = 0.1$ and for some values φ_0 and β we find $q_*(\varphi_0, \beta)$ such that for $q < q_*(f_0, \beta)$ the limiting equality

$$\varphi(t) \to 0 \quad \text{at} \quad t \to \infty, \tag{13}$$

is valid, whereas in the opposite case $q > q_*(f_0, \beta)$ Eq.(13) is not fulfilled. The boundary $q_*(\varphi_0, \beta)$ depends on the initial phase β which is unknown in the general case. That is why we introduce two attraction basins in the plane of parameters (φ_0, q)

$$\begin{aligned} G_a(\varphi_0): &\quad q < q_*^-(\varphi_0), & q_*^-(\varphi_0) &= \min_{\beta\in[0,2\pi)} q_*(\varphi_0,\beta), \\ G_p(\varphi_0): &\quad q_*^-(\varphi_0) < q < q_*^+(\varphi_0), & q_*^+(\varphi_0) &= \max_{\beta\in[0,2\pi)} q_*(\varphi_0,\beta). \end{aligned} \tag{14}$$

see Fig. 3 Eq.(13) is fulfilled for all values β In basin G_a; it is fulfilled only for some values β in basin G_p, and it is newer fulfilled in part G_0 of plane (φ_0, q).

The boundaries $q_*^-(\varphi_0)$ and $q_*^+(\varphi_0)$ are numerical solutions of the exact problem (2), (6). The approximate problem (11), (12) gives the close results (the correspond-

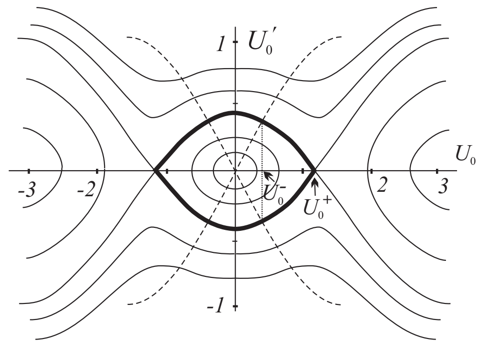


Fig. 4 The attraction basin in the phase plane (U_0, U_0') at $q = 0.3$

ing curve $q_*^-(\varphi_0)$ in Fig. 3 is shown as a dashed line, and the difference between the exact and approximate curves is so small that it is impossible to see it in figure $q_*^+(\varphi_0)$).

Eq.(11) is convenient for qualitative analysis in the phase plane (U_0, U_0'). The trajectories $U_0(\theta), U_0'(\theta)$ for $q = 0.3$, $n = 0$ are shown in Fig. 4. A bold curve separates the attraction basin while the possible values $|U_0'| \leq |\sin U_0|$ are marked by dashed lines.

2.3 Attraction basins for Kapitsa's problem at random excitation

Let the vertical support vibration $x_e(t) = \xi(t)$ be random (Fig. 1a), and $\xi(t)$ be a stationary process with zero excitation and spectral density $S_\xi(\lambda)$. We consider the problems of Section 2 for the case of random excitation. Eq.(1) reads as:

$$J\frac{d^2\varphi}{d\tilde{t}^2} + n_1\frac{d\varphi}{d\tilde{t}} - \frac{mL}{2}\left(g + \frac{d^2\tilde{\xi}}{d\tilde{t}^2}\right)\sin\varphi = 0. \tag{15}$$

We rewrite Eq.(15) in the dimensionless form, relating time $\tilde{t}$ to $1/\omega$ (ω is the typical frequency of vibration of support), and relating excitation $\tilde{\xi}(\tilde{t})$ to the average amplitude of support vibration $\sigma_{\tilde{\xi}}$:

$$\ddot{\varphi} + \varepsilon n\dot{\varphi} - \left(\varepsilon^2 q + \varepsilon\ddot{\xi}\right)\sin\varphi = 0, \tag{16}$$

where derivative with respect to t is denoted by a dot, and

$$t = \omega\tilde{t}, \quad \tilde{\xi}(\tilde{t}) = \sigma_{\tilde{\xi}}\xi(t), \quad \sigma_{\tilde{\xi}}^2 = \int_{-\infty}^{\infty} S_{\tilde{\xi}}(\tilde{\lambda})d\tilde{\lambda}, \quad \varepsilon = \frac{3\sigma_{\tilde{\xi}}}{2L}, \quad q = \frac{3Lg}{2\sigma_{\tilde{\xi}}^2\omega^2}. \tag{17}$$

Here ε is a small parameter that is proportional to the average amplitude of the support vibration $\sigma_{\tilde{\xi}}$ and $\xi(t)$ is the normalized process with a unit dispersion. The spectral densities and the dispersions of $\xi(t)$ and its derivatives are as follows:

$$S_{\xi}(\lambda) = \frac{S_{\tilde{\xi}}(\tilde{\lambda}\omega)}{\sigma^2_{\tilde{\xi}}}, \quad S_{\dot{\xi}}(\lambda) = \lambda^2 S_{\xi}(\lambda), \quad S_{\ddot{\xi}}(\lambda) = \lambda^4 S_{\xi}(\lambda), \tag{18}$$

$$\sigma^2_{\dot{\xi}} = \int_{-\infty}^{\infty} S_{\xi}(\lambda)\lambda^2 d\lambda.$$

We solve Eq.(16) with the initial conditions $\varphi(0) = \varphi_0,\ \dot{\varphi}(0) = 0$, and use two ways for solving the problem.

One of them is a statistical simulation [15, 11]. We model a random process $\xi(t)$ as a sum of harmonic summands with random amplitudes and phases. For this aim we choose Λ so that the part of frequencies $\lambda > \Lambda$ can be neglected and divide the interval $0 \le \lambda \le \Lambda$ by points $\lambda_n,\ n = 1, \ldots, N$. Then the approximate realization of a random process $\xi(t)$ read as:

$$\xi(t) = \sum_{n=1}^{N} p_n(\eta_n \cos(\hat{\lambda}_n t) + \kappa_n \sin(\hat{\lambda}_n t)), \tag{19}$$

$$p_n = \sqrt{2S_{\xi}(\hat{\lambda}_n)(\lambda_n - \lambda_{n-1})}, \qquad \hat{\lambda}_n = (\lambda_n + \lambda_{n-1})/2,$$

where η_n and κ_n are the random independent standard Gaussian numbers ($\mathbf{E}\eta_n = \mathbf{E}\kappa_n = 0,\ \mathbf{E}\eta_n^2 = \mathbf{E}\kappa_n^2 = 1$, and $\mathbf{E}$ denotes expectation). Then a numerical solution of Eq.(16) with the initial conditions (6) gives a realization of a random process $\varphi(t)$.

As an example, we consider random process $\tilde{\xi}(\tilde{t})$ with the spectral density

$$S_{\tilde{\xi}}(\tilde{\lambda}) = \frac{\tilde{c}}{(\tilde{\lambda}^4 + 2(\tilde{\alpha}^2 - \omega^2)\tilde{\lambda}^2 + (\tilde{\alpha}^2 + \omega^2)^2)(\tilde{\lambda}^2 + \omega^2)}. \tag{20}$$

According to Eqs.(17),(18), for the dimensionless process $\xi(t)$ the spectral density reads as:

$$S_{\xi}(\lambda) = \frac{c}{(\lambda^4 + 2(\alpha^2 - 1)\lambda^2 + (\alpha^2 + 1)^2)(\lambda^2 + 1)}, \quad \lambda = \tilde{\lambda}/\omega, \quad \alpha = \tilde{\alpha}/\omega, \tag{21}$$

where constant c is to be found from the condition $\sigma^2_{\xi} = \int_{-\infty}^{\infty} S_{\xi}(\lambda)d\lambda = 1$. We find $\sigma^2_{\dot{\xi}} = (1 + \tilde{\alpha}^2)/(1 + 2\alpha)$. The constant $\tilde{c}$ is introduced so that the value of $\sigma^2_{\tilde{\xi}}$ and small parameter ε can be taken arbitrary.

We take the following values: $\varepsilon = 0.01,\ n = 0.1,\ \alpha = 0.2,\ N = 200$ and consider the case $\varphi_0 > 0$. The spectral density $S_{\xi}(\lambda)$ of the normalized process $\xi(t)$ is plotted in Fig. 5. The maximum of $S_{\xi}(\lambda)$ is close to $\lambda = 1$ and $\sigma_{\xi} = 1$. The attraction basins G_a and G_p obtained by a numerical solution of Eq.(16) are shown in Fig. 5. In each numerical experiments we take 10 independent realizations of process $\varphi(t)$. A point (φ_0, q) is included in G_p if at least one realization converges to zero at $t \to \infty$, and at least one realization converges to $\pm\pi$. Hence in the areas G_p and G_0 all 10 realizations tends to zero and to $\pm\pi$ at $t \to \infty$, respectively. The boundaries of G_p are denoted by $\hat{q}^-$ and $\hat{q}^+$.

The second way of analysis of Eq.(16) is applying the two-scale expansion (7):

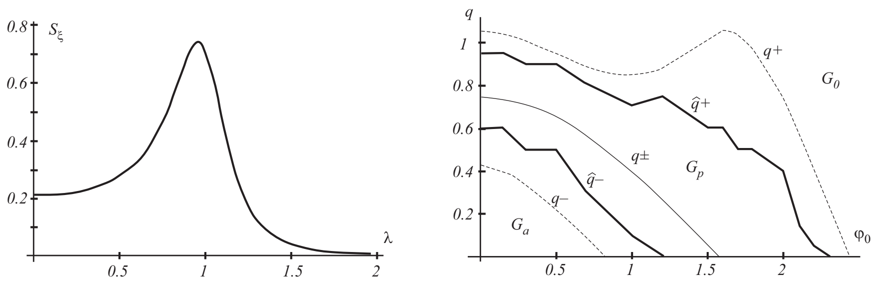


Fig. 5 Spectral dencity (left); Attraction basins (right)

$$\varphi(t,\theta,\varepsilon) = U(\theta,\varepsilon) + V(t,\theta,\varepsilon), \tag{22}$$

$$U(\theta,\varepsilon) = \sum_{m=0}^{\infty} U_m(\theta)\varepsilon^m, \quad V(t,\theta,\varepsilon) = \sum_{m=0}^{\infty} V_m(t,\theta)\varepsilon^m,$$

where the average value V is equal to zero

$$\langle V \rangle = \frac{1}{T}\int_0^T V(t,\theta,\varepsilon)dt = 0, \quad T = O(\varepsilon^{-1}). \tag{23}$$

Repeating the calculations of Subsect. 2.2, we successively obtain:

$$V_0(t,\theta) = 0, \quad V_1(t,\theta) = \xi(t)\sin U_0, \quad \frac{\partial^2 V_2}{\partial t^2} + H(t,\theta) = 0, \tag{24}$$

with

$$H = 2\frac{\partial^2 V_1}{\partial t \partial \theta} + \frac{d^2 U_0}{\partial \theta^2} + n\frac{dU_0}{\partial \theta} - q\sin U_0 - (U_1 + V_1)\frac{d^2\xi}{dt^2}\cos U_0. \tag{25}$$

The condition $\langle H \rangle = 0$ leads to equation for function $U_0(\theta)$

$$U_0'' + nU_0' + (\chi\cos U_0 - q)\sin U_0 = 0, \quad \chi = -\left\langle \xi(t)\ddot{\xi}(t) \right\rangle, \quad U_0(0) = \varphi_0, \tag{26}$$

The second initial condition $\dot{\varphi}(0) = 0$ due to Eq.(8) and Eq.(24) yields

$$U_0'(0) = -\dot{\xi}(0)\sin\varphi_0. \tag{27}$$

The problem (26), (27) contains two random values: χ and $\dot{\xi}(0)$, and we use this problem to estimate the attraction basins. To construct them we note that for Gaussian values with a probability 0.95 the following inequalities are valid:

$$\mathbf{E}(\chi) - 2\sigma_\chi \le \chi \le \mathbf{E}(\chi) + 2\sigma_\chi, \quad -2(\mathbf{E}(\chi) - 2\sigma_\chi)^{1/2} \le \dot{\xi}(0) \le 2(\mathbf{E}(\chi) + 2\sigma_\chi)^{1/2}, \tag{28}$$

where $\mathbf{E}(\chi)$ is the expectation of χ and σ_χ is the root-mean-square.

For the taken values of the random process (19) we obtain

$$\mathbf{E}(\chi) \approx -\frac{1}{T}\int_0^T \xi(t)\ddot{\xi}(t)dt \approx \frac{1}{T}\int_0^T \left(\dot{\xi}(t)\right)^2 dt \approx \sigma_{\dot{\xi}}^2 = 0.743. \tag{29}$$

From Eq.(19) we have:

$$\chi \approx \frac{1}{2}\sum_{n=1}^{N} p_n \hat{\lambda}_n^2(\eta_n^2 + \kappa_n^2). \tag{30}$$

Taking a large number (say, 10000) of random sets $(\eta_n,\ \kappa_n, n = 1, \ldots, N)$ and using (30) we obtain the following value of root-mean-square $\sigma_\chi = 0.157$.

We put the upper and lower bounds of values of χ and $\dot{\varphi}(0)$ in Eqs.(28) and obtain from Eq.(26) the boundaries $q^-(\varphi_0)$ and $q^+(\varphi_0)$ of attraction basins which are shown in Fig. 5. For comparison, curve $q^\pm(\varphi_0)$ corresponding to the values $\chi = \sigma_{\dot{\xi}}^2,\ \dot{\xi}(0) = 0$ is also given there.

In particular, it follows from Eq.(26) that the vertical position (with a probability 0.95) is stable provided that

$$q < \sigma_{\dot{\xi}}^2 - 2\sigma_\chi, \tag{31}$$

and for taken values if $q < 0.429$.

2.4 A Kapitsa's pendulum on the flexible support

Let us consider a rigid rod with an elastically supported lower end, see Fig. 1b. In terms of the dimensionless variables (3) the motion of rod on vibrating support is described by the equation

$$\ddot{\varphi} + n\alpha\dot{\varphi} + \alpha^2(b\varphi - q\sin\varphi) + \alpha\sin\varphi\sin(t+\beta) = 0, \quad b = \frac{4b_0 L}{3ma^2\omega^2}. \tag{32}$$

In addition to Eq.(2) describing the classic Kapitsa's pendulum, the bending support stiffness b_0 is introduced.

At $b < q$ and $a = 0$ the vertical rod position is unstable. The rod is stable at $\varphi = \varphi_0$, where φ_0 is the root of equation

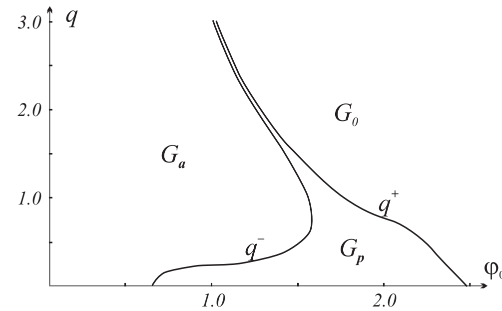
$$b\varphi_0 = q\sin\varphi_0, \quad \text{or} \quad b = kq, \quad k = \frac{\sin\varphi_0}{\varphi_0} < 1. \tag{33}$$

Now we seek the conditions ensuring stable vertical position in the presence of support vibration.

We seek a solution of Eq.(32) satisfying the initial conditions $\varphi(0) = \varphi_0, \dot{\varphi}(0) = 0$. We assume that the angle $\varphi_0 < \pi$ is a leading parameter of problem, and a stiffness parameter $b = kq$. As in Subsections 2.2, 2.3, we use two-scale expansions, that in the first approximation for a slowly changing function $U_0(\theta)$ lead to Cauchy problem:

$$U_0'' + nU_0' + F(U_0) = 0, \quad U_0(0) = \varphi_0, \quad U_0'(0) = -\sin\varphi_0\sin\beta, \tag{34}$$

Fig. 6 Attraction basins in the plane (φ_0, q)



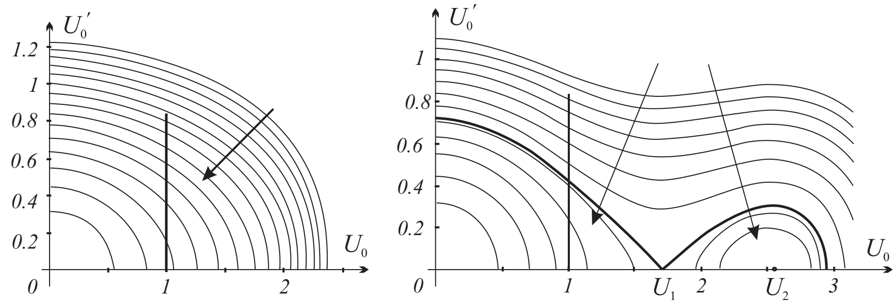


Fig. 7 A phase plane at $f_0 = 1$, and at $q = 1$ (left) and $q = 0.1$ (right)

with $F(U_0) = kqU_0 + ((1/2)\cos U_0 - q)\sin U_0$. At $n > 0, q < q_*^+ = 1/(2(1-k))$ the solution $U_0(\theta) \equiv 0$ is asymptotically stable. As in Subsect. 2.2, at $q < q_*^+$ we seek an attraction basin of this solution. A plane of parameters (φ_0, q) consists of three parts G_a, G_p, G_0 and for $\varepsilon = 0.01,\ n = 0.1,\ q \leq 3$ it is shown in Fig. 6. At $q > 3$ the boundaries q^- and q^+ coincide, and $q^- \approx q^+ \approx q_*^+$.

At $n = 0$ the trajectories

$$\dot{U}_0^2 + 2\int_0^{U_0} F(U)dU = C, \tag{35}$$

(with arbitrary constant C) in a phase plane (U_0, U_0') are symmetric with respect to axes OU_0 and OU_0', and we consider a quarter part of plane $U_0, U_0' \geq 0$. At $n > 0$ a point passes from one trajectory to another with the lower value of C. In Fig. 7 the phase planes for $\varphi_0 = 1$ and for two values $q = 1$ and $q = 0.1$ are shown. The direction of decreasing values of C is indicated by arrow. A set of possible values of $|U_0'|$ is marked by a vertical line. For $q = 1$ all $|U_0'|$ lie in the attraction basin of point $U_0 = U_0' = 0$, therefore, for all values of the initial phase β Eq.(13) is fulfilled, and $(\varphi_0, q) \in G_a$. For $q = 0.1$ only a part of values $|U_0'|$ lie in the attraction basin of point $U_0 = U_0' = 0$ (that is separated by a bold line), and as a result we have $(\varphi_0, q) \in G_p$.

3 Generalized Kapitsa's Pendulum. Flexible Rod

We consider a flexible inverted pendulum in the case of a harmonic vertical vibration of the support $z_0(t) = a\sin(\omega t + \beta)$ (see also [2, 10]). Small transverse oscillations of a longitudinally compressed flexible rod of length L about the vertical position in the coordinate frame of the support are described by the equation

$$D\frac{\partial^4 w}{\partial x^4} + \frac{\partial}{\partial x}\left(P_s\frac{\partial w}{\partial x}\right) + \rho S\frac{\partial^2 w}{\partial t^2} = 0, \quad P_s = P_w(x) + P_v(x,t). \tag{36}$$

Here $w(x,t)$ is the deflection, $D = EI$ is the bending rigidity, ρ is the material density, S is the cross-sectional area. The upper end $x = L$ of the rod is free ($w_{xx} = w_{xxx} = 0$), and the lower end is clamped ($w = w_x = 0$).

The axial force P_s is assumed to have two summands: $P_w(x) = P(L-x)/L$ with $P = \rho gSL$ is caused by weight of the rod, and $P_v(x,t)$ is due to the support vibration. For the inextensible rod $P_v(x,t) = -\rho a\omega^2 S(L-x)\sin(\omega t+\beta)$, and for the extensible rod the axial force $P_v(x,t)$ is determined by the propagating longitudinal waves, see [2, 10] for detail:

$$P_v(x,t) = -ES\frac{\partial u}{\partial x} = -ES(a/v)(\cos v\hat{x}\,\mathrm{tg}\,v - \sin v\hat{x})\sin(\omega t + \beta), \tag{37}$$

with $\hat{x} = x/L,\ v = L\omega/c,\ c^2 = E/\rho$. Here E, ρ, c, ω, v are the Young modules, the mass density, the sound velocity in the rod, the support frequency and the dimensionless frequency, respectively. For the inextensible rod $v = 0$.

The conditions for stability of the vertical position of the rod subjected to the support vibrations was found in [2, 10] for both inextensible and extensible flexible rods. Our aim is to obtain the attraction basin of this problem, but at first we repeat some results of papers [2, 10].

Equation (36) in the dimensionless form is given by:

$$\frac{\partial^4 w}{\partial \hat{x}^4} + P_*\frac{\partial}{\partial \hat{x}}\left((1 - \hat{x} - g_a p_v(\hat{x})\sin(\hat{t}+\beta))\frac{\partial w}{\partial \hat{x}}\right) + P_* g_L\frac{\partial^2 w}{\partial \hat{t}^2} = 0, \quad P_* = \frac{P_0 L^2}{D}, \tag{38}$$

with $\hat{t} = \omega t,\ g_a = a\omega^2/g,\ g_L = L\omega^2/g$. For an extensible rod $p_v(\hat{x}) = (\cos v\hat{x}\,\mathrm{tg}\,v - \sin v\hat{x})/v$, and for the inextensible rod $p_v(\hat{x}) = 1 - \hat{x}$. The last expression follows from the previous one at $v \to 0$.

In what follows we omit a sign $\hat{}$.

The solution of Eq.(38) is sought in the form of a series

$$w(x,t) = \sum_{k=1}^{N}\Psi_k(x)w_k(t), \tag{39}$$

where $\Psi_k(x)$ are eigenfunctions of the boundary-value problem

$$\frac{d^4\Psi}{dx^4}+\lambda\frac{d}{dx}\left((1-x)\frac{d\Psi}{dx}\right)=0,\quad \Psi(0)=\Psi_x(0)=\Psi_{xx}(1)=\Psi_{xxx}(1)=0. \tag{40}$$

This problem relates to the problem of static equilibrium bifurcation of a heavy rod with a free upper end and a clamped lower end. The solution of Eq.(40) may be expressed in terms of the Airy functions [1], and the first eigenvalues are $\lambda_1 = 7.8373,\ \lambda_2 = 55.98,\ \lambda_3 = 148.5,\ \lambda_4 = 285.4$. For $P_* > \lambda_1$ the rod buckles due to the gravity.

Due to the orthogonality relation $\int_0^1 (1-x)\Psi_k'\Psi_n'\,dx = 0$ for functions $\Psi_k(x)$, we obtain the system for unknown functions $w_k(t)$:

$$\begin{array}{l}\sum_{k=1}^N a_{nk}\dfrac{d^2w_k}{dt^2}+\varepsilon\left(\dfrac{b_n}{g_a}\left(\dfrac{p_n}{P_*}-1\right)+c_n\sin(t+\beta)\right)w_n=0,\quad n=1,...,N,\\ \varepsilon=\dfrac{a}{L}\ll 1,\end{array} \tag{41}$$

where $a_{kn} = \int_0^1 \Psi_k\Psi_n dx,\quad b_n = \int_0^1 (1-x)(\Psi_n')^2dx,\quad c_n = \int_0^1 p_v(x)(\Psi_n')^2dx$, and for the inextensible rod $c_n = b_n$.

The first coefficients are $a_{11} = 0.128,\ b_1 = 0.202$. For the extensible rod the coefficient c_1 depends on ν and $|c_1(\nu)|$ is plotted in Fig. 8. At $\nu = \pi/2 + n\pi,\ n = 0, 1, ...,$ the value $c_1(\nu) \to \infty$, that corresponds to resonances of longitudinal vibration of the rod.

3.1 Conditions of stability of the vertical position

The introduced small parameter ε allows us to use two-scale expansions. We put $g_a = \zeta/\varepsilon$ and write Eq. (41) in the matrix form:

$$\mathbf{A}\cdot\frac{d^2\mathbf{W}}{dt^2}+\frac{\varepsilon^2}{\zeta}\mathbf{P}\cdot\mathbf{W}+\varepsilon\mathbf{C}\cdot\mathbf{W}\sin t=0, \tag{42}$$

where $\mathbf{W} = \{w_k\}^T_{k=1,N}$ is the vector of unknown functions, $\mathbf{A} = \{a_{kn}\}_{k,n=1,N}$ is the symmetric matrix, $\mathbf{P}$ and $\mathbf{C}$ are the diagonal matrices with elements $\{b_k(\lambda_k/P_* - 1)\}_{k=1,N}$ and $\{b_k\}_{k=1,N}$, respectively.

Similar to Sect. 2, we look for the unknown function $\mathbf{W} = \mathbf{W}(t,\theta,\varepsilon),\ \theta = \varepsilon\, t$ in the form:

$$W(t,\theta,\varepsilon)=\sum_{m=0}^{\infty}(\mathbf{U}_m(\theta)+\mathbf{V}_m(t,\theta))\varepsilon^m,\qquad \int_0^{2\pi}\mathbf{V}_m(t,\theta))dt=0,\quad m=0,1,\ldots \tag{43}$$

Then from Eq. (42) we obtain consecutively:

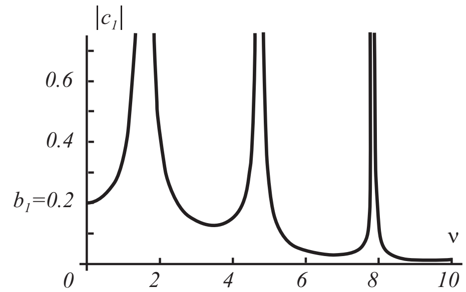


Fig. 8 Schematic of coefficient $|c_1(\nu)|$

Table 1 Dependence of approximations $\zeta_*^{(1)}$ and $\zeta_*^{(2)}$ on P_*

P_*	$\le \lambda_1$	8	10	25	50	100	∞
$\zeta_*^{(1)}$	0	0.02	0.27	0.87	1.06	1.16	1.26
$\zeta_*^{(2)}$	0	0.02	0.25	0.85	1.06	1.17	1.29

$$\begin{aligned} &\mathbf{V}_0(t,\theta) \equiv 0, \qquad \mathbf{V}_1(t,\theta) = \mathbf{A}^{-1}\cdot\mathbf{C}\cdot\mathbf{U}_0 \sin t, \\ &\mathbf{A}\cdot\frac{d^2\mathbf{U}_0}{d\theta^2} + \mathbf{D}\cdot\mathbf{U}_0 = 0, \qquad \mathbf{D} = \frac{\mathbf{P}}{\zeta} + \frac{1}{2}\,\mathbf{C}\cdot\mathbf{A}^{-1}\cdot\mathbf{C}. \end{aligned} \tag{44}$$

It follows from Eq.(44) that the vertical position of rod is stable if matrix $\mathbf{D}$ is positively definite that allows us to find the critical value ζ_* of the loading parameter $\zeta = a^2\omega^2/(Lg)$.

For the single mode approximation ($N = 1$)

$$\zeta_* = \frac{2a_{11}b_1}{c_1^2}\left(1 - \frac{\lambda_1}{P_*}\right) = \frac{0.0517}{c_1^2}\left(1 - \frac{7.84}{P_*}\right), \tag{45}$$

and for the inextensible rod

$$\zeta_* = \frac{2a_{11}}{b_1}\left(1 - \frac{\lambda_1}{P_*}\right) = 1.27\left(1 - \frac{7.84}{P_*}\right). \tag{46}$$

Calculation of matrix $\mathbf{D}$ shows that the single mode approximation gives an acceptable accuracy for the critical value ζ_*. The single mode ($\zeta_*^{(1)}$) and the two-mode ($\zeta_*^{(2)}$) critical values of ζ are given for some values of P_* in Table 1 for the inextensible rod.

Remark 1. The hinged support ($w = w_{xx} = 0$ at $x = 0$) of lower end is also studied in Refs. [2, 10]. In this case $\lambda_1 = 0,\ \lambda_2 = 25.64,\ a_{11} = 1/3,\ b_1 = 0.5$, and Eq. (46) gives $\zeta_* = 4/3$ independently of the value P_* that exactly corresponds to the critical value ($q = 1$) for a rigid rod (see Sect. 2). The higher approximations in Eq.(39) show that the value ζ_* slightly exceeds $\zeta_* = 4/3$, namely $\zeta_* = 1.37$ at $P_* = 120$. Dependence $\zeta_*(P_*)$ for the hinged and clamped lower end of rod are shown in Fig. 9.

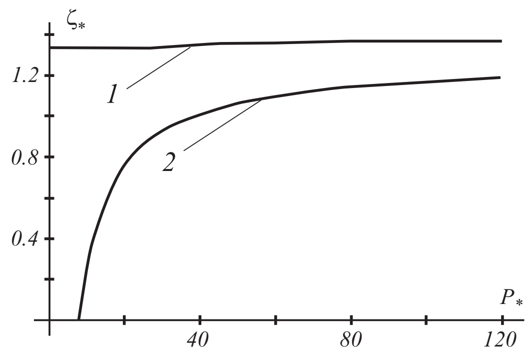


Fig. 9 Functions $\zeta_*(P_*)$ for a hinged (*1*) and for a clumped (*2*) lower end of rod

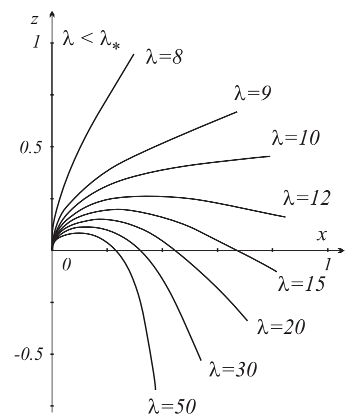


Fig. 10 Equilibrium forms of buckled rod

For extensible rod, the inequality $|c_1(\nu)| > b_1$ is valid at some intervals of parameter ν (see Fig. 8 with $|c_1(0)| = b_1$), therefore according to Eqs. (45) and (46) the influence of longitudinal waves in the rod tends to decrease the critical level ζ_* of the support vibrations. A numerical example is presented in [10].

In Ref. [14], stationary positions were found and the stability was investigated for a flexible Chelomei pendulum under the support vibration and the used investigation methods were close to ours. However in contrast to Eq. (36), the axial compressive force caused by the rod weight is not taken into account when describing the flexural deformation of the rod. That is why no benchmark of the results was made.

3.2 On the attraction basins for a flexible rod

We consider a vertical flexible inextensible rod with a clamped lower end and a free upper end, cf. [9]. The long rod buckles under weight at $P_* > \lambda_* = 7.84$ (see Fig. 10), however the rod takes again the stable vertical position for high level of the support vibration, namely at $\zeta > \zeta_*$, see Eq.(45). Now we seek the attraction basin of this position.

Motion of the extensible rod under weight and vertical support vibrations is described by the equilibrium equations [13]:

$$
\begin{aligned}
&D\frac{\partial\varphi}{\partial s} = M(s,t) = \int_s^L (F_x(s_1)(z(s_1) - z(s)) - F_z(s_1)(x(s_1) - x(s)))ds_1,\\
&E\, e(s,t) = \sin\varphi(s)\int_s^L F_x(s_1)ds_1 + \cos\varphi(s)\int_s^L F_z(s_1)ds_1,
\end{aligned}
\tag{47}
$$

where

$$
\begin{aligned}
&x(s) = \int_0^s \sin\varphi(1+e)\,ds, \quad z(s) = \int_0^s \cos\varphi(1+e)\,ds,\\
&F_x = -S\rho\ddot{x}, \quad F_z = -S\rho(g + a\omega^2\sin(\omega t + \beta) + \ddot{z}).
\end{aligned}
\tag{48}
$$

Here s $(0 \le s \le L)$ is the length of arc of the rod axis, $\varphi(s,t)$ is the angle between tangent to the rod axis and the vertical and $e(s,t)$ is the longitudinal deformation of the rod axis.

For the inextensible rod $e = 0$.

In the static case (with $a = 0$) and for inextensible rod we can simplify Eq. (47) to obtain the boundary-value problem

$$
\frac{d^2\varphi}{ds^2} + P_*(1-s)\sin\varphi = 0, \quad \varphi(0) = 0,\ \varphi'(1) = 0,
\tag{49}
$$

where s is related to L. The forms of buckled rod shown in Fig. 10 are obtained from Eq. (49).

For the approximate analysis of attraction basin in the neighborhood of vertical position of the rod we use the single mode approximations for unknown functions $\varphi(s,t)$ and $e(s,t)$:

$$
\begin{aligned}
&\varphi(s,t) = \Phi(s)u(t), \quad \Phi(s) = s - 0.200383s^2 - 0.81018s^3 + 0.457827s^4,\\
&e(s,t) = e(s)v(t), \quad e(s) = 1 - s, \qquad 0 \le s \le 1.
\end{aligned}
\tag{50}
$$

Function $e(s) = 1 - s$ yields the first natural frequency 1.58 of longitudinal vibration instead of the exact value $\pi/2 = 1.57$. Function $\Phi(s)$ is close to the first eigenfunction $\Phi_1(x)$ of problem (40).

Table 2 displays the exact coordinates x^e, z^e of the rod end $s = 1$ obtained from Eq.(49). They are compared with the approximate values x^a, z^a from approximation (50) and shown in Fig. 10. Here u_* corresponds to the equilibrium state of vibration-free rod at given P_*. In what follows we will use approximation (50) at $|u| \le 6$.

Remark 2. The other possibility, that is not used here, consists in replacing a flexible rod by a system of N rigid rods connected by elastic angular strings (Fig. 11). For the Kapitsa problem the case $N = 1$ is considered in [9, 16] where an attraction basin is constructed.

Assumption (50) suggests the rod to be reduced to a system with two degrees of freedom, therefore we use the Lagrange equations of the second kind. We introduce the designations

Table 2 Exact and approximate coordinates of the upper rod end

P_*	u_*	x^e	x^a	z^e	z^a
8	0.942	0.298	0.293	0.946	0.947
9	2.432	0.674	0.672	0.667	0.668
10	3.206	0.793	0.788	0.456	0.457
11	3.760	0.838	0.829	0.292	0.292
12	4.195	0.850	0.836	0.162	0.181
13	4.555	0.845	0.825	0.057	0.054
14	4.863	0.831	0.804	−0.030	−0.034
15	5.133	0.812	0.777	−0.102	−0.108

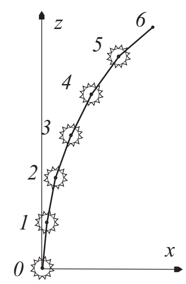


Fig. 11 Discrete model of a flexible rod

$$\{s, x, z, a\} = L\{\hat{s}, \hat{x}, \hat{z}, \varepsilon\}, \quad \omega t = \hat{t},$$
$$P_* = \frac{PL^2}{D}, \quad g_L = \frac{L\omega^2}{g}, \quad \hat{T} = \frac{T}{PL}, \quad \hat{\Pi} = \frac{\Pi}{PL}, \tag{51}$$

and then omit a sign $\hat{}$. Here T and Π are the kinetic and potential energy

$$T = \frac{g_L}{2}\int_0^1 \left(\dot{x}^2 + (\dot{z} + \varepsilon\cos(t+\beta))^2\right) ds, \quad (\,)' = \frac{d(\,)}{du},$$
$$\Pi = \int_0^1 \left(\frac{(\varphi')^2}{2P_*} ds + z + \frac{c^2 e^2}{2gL}\right) ds. \tag{52}$$

where $x(s,t)$ and $z(s,t)$ are given by Eq.(48).

3.3 Lagrange equations of the second kind

At first, we find from Eqs.(48) and (50)

$$x(s,t) = X(u,s) + vX_1(u,s), \quad z(s,t) = Z(u,s) + vZ_1(u,s) \tag{53}$$

with

$$X=\sum_{k=0}^{K}\frac{(-1)^k u^{2k+1}I_{2k+1}}{(2k+1)!},\quad X_1=\sum_{k=0}^{K}\frac{(-1)^k u^{2k+1}J_{2k+1}}{(2k+1)!},\quad Z=\sum_{k=0}^{K}\frac{(-1)^k u^{2k}I_{2k}}{(2k)!},$$

$$Z_1=\sum_{k=0}^{K}\frac{(-1)^k u^{2k}J_{2k}}{(2k)!},\quad I_n(s)=\int_0^s\Phi^n(\sigma)d\sigma,\quad J_n(s)=\int_0^s\Phi^n(\sigma)e(\sigma)d\sigma.$$

and

$$\dot{x}(s,t)=(X'(u,s)+vX_1'(u,s))\dot{u}+X_1(u,s)\dot{v},$$
$$\dot{z}(s,t)=(Z'(u,s)+vZ_1'(u,s))\dot{u}+Z_1(u,s)\dot{v}.$$

Now the kinetic and the potential energy are as follows:

$$T=g_L\left[\left(\frac{F_{00}}{2}+F_{01}v+\frac{F_{02}v^2}{2}\right)\dot{u}^2+(F_{10}+F_{11}v)\dot{u}\dot{v}+\frac{F_{20}\dot{v}^2}{2}+\varepsilon G\cos(t+\beta)\right],$$
$$G=(G_{00}+G_{01}v)\dot{u}+G_{10}\dot{v},$$
$$\Pi=\frac{c_0u^2}{2P_*}+\int_0^1(Z+vZ_1)ds+\frac{c^2v^2}{2gL}\int_0^1 e^2ds,\qquad c_0=\int_0^1(\Phi')^2ds=0.3184 \tag{54}$$

with

$$F_{00}=\int_0^1(X'^2+Z'^2)ds,\quad F_{01}=\int_0^1(X'X_1'+Z'Z_1')ds,$$
$$F_{10}=\int_0^1(X'X_1+Z'Z_1)ds,\quad F_{11}=\int_0^1(X_1'X_1+Z_1'Z_1)ds,$$
$$F_{02}=\int_0^1(X_1'^2+Z_1'^2)ds,\quad F_{20}=\int_0^1(X_1^2+Z_1^2)ds,$$
$$G_{00}=\int_0^1 Z'ds,\quad G_{01}=\int_0^1 Z_1'ds,\quad G_{10}=\int_0^1 Z_1ds.$$

The Lagrange equations read as:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{u}}\right)-\frac{\partial\mathcal{L}}{\partial u}=0,\quad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{v}}\right)-\frac{\partial\mathcal{L}}{\partial v}=0,\qquad \mathcal{L}=T-\Pi. \tag{55}$$

We seek solutions of Eqs. (55) by using two-scale expansions, and we keep only the following first terms:

$$\begin{aligned}&u(t)=U(\theta)+\varepsilon u_1(\theta,t)+\varepsilon^2u_2(\theta,t),\quad v(t)=v_0(\theta)+\varepsilon v_1(\theta,t),\\&\langle u_1\rangle=\langle u_2\rangle=\langle v_1\rangle=0.\end{aligned} \tag{56}$$

where $\theta=\varepsilon t$ is the slow time. Then the derivatives with respect to time are given by:

$$\dot{u}=\varepsilon\dot{u}_1+\varepsilon U_{,\theta}+\varepsilon^2\dot{u}_2+O(\varepsilon^3),\qquad \ddot{u}=\varepsilon\ddot{u}_1+\varepsilon^2U_{,\theta\theta}+\varepsilon^2\ddot{u}_2+2\varepsilon^2\dot{u}_{1,\theta}+O(\varepsilon^3).$$

Equations in (55) are cumbersome. In the first and second equations in Eq. (55) we omit the terms with orders smaller than ε^2 and ε, respectively. Then we have

$$\begin{aligned}
&g_L\,[\, F_{00}\ddot{u} + 2F_{01}(\dot{u}\dot{v}+\ddot{u}v) + F'_{00}\dot{u}^2/2 + F_{10}\ddot{v} - F'_{20}\dot{v}^2/2 - G_{00}\varepsilon\sin(t+\beta)\,] + \\
&\qquad + \frac{c_0 u}{P_*} + G_{00} = 0, \\
&g_L[F_{10}\ddot{u} + F_{20}\ddot{v} - G_{10}\varepsilon\sin(t+\beta)] + G_{10}(u) + \frac{c^2 v}{3gL} = 0.
\end{aligned} \tag{57}$$

Here all the functions F_{ij}, G_{ij} depend on u.

We find $v_0 = -3gL/c^2 G_{10}(U_0)$ and put $u_1 = \hat{u}_1(\theta)\sin(t+\beta)$, $\quad v_1 = \hat{v}_1(\theta)\sin(t+\beta)$. Then the terms of order ε in Eq. (57) give equations for functions $\hat{u}_1(\theta),\ \hat{v}_1(\theta)$:

$$\begin{aligned}
&F_{00}(U)\hat{u}_1 + F_{10}(U)\hat{v}_1 + G_{00}(U) = 0, \\
&\nu^2\,(F_{10}(U)\hat{u}_1 + F_{20}(U)\hat{v}_1 + G_{10}(U)) - \hat{v}_1/3 = 0, \quad \nu = \frac{L\omega}{c}.
\end{aligned} \tag{58}$$

In what follows the sign $\hat{}$ is omitted.

After time averaging the terms of order ε^2 in the first equation in Eq.(57) give the equation for function $U(\theta)$:

$$\begin{aligned}
&\zeta\left(F_{00}U_{,\theta\theta} + 1/2F'_{00}\left(U^2_{,\theta} - 1/2u_1^2\right) - 1/2F'_{10}u_1v_1 - 1/4F'_{20}v_1^2 - 1/2G'_{00}u_1\right) + \\
&\qquad + \frac{c_0 U}{P_*} + G_{00} = 0
\end{aligned} \tag{59}$$

with $\zeta = \varepsilon^2 g_L$. Here all the functions F_{ij} and G_{00} depend on U.

Equation (59) is the principle equation for following analysis of the attraction basin. We rewrite it in the form:

$$\begin{aligned}
&F_{00}U_{,\theta\theta} + 1/2F'_{00}U^2_{,\theta} + nU_{,\theta} + F(U) = 0, \\
&F(U) = -1/4F'_{00}u_1^2 - 1/2F'_{10}u_1v_1 - 1/4F'_{20}v_1^2 - 1/2G'_{00}u_1 + \zeta^{-1}\left(\frac{c_0 U}{P_*} + G_{00}\right),
\end{aligned} \tag{60}$$

where the resistance term $nU_{,\theta}$ is introduced.

3.4 Attraction basins for inextensible rod

For the inextensible rod we put $\nu = 0$, $v_1 = 0$, $u_1 = -G_{00}/F_{00}$ in the previous formulas, then Eq.(60) is as follows

$$F(U) = -1/4\,F'_{00}u_1^2 - 1/2\,G'_{00}u_1 + \zeta^{-1}\left(\frac{c_0 U}{P_*} + G_{00}\right) \tag{61}$$

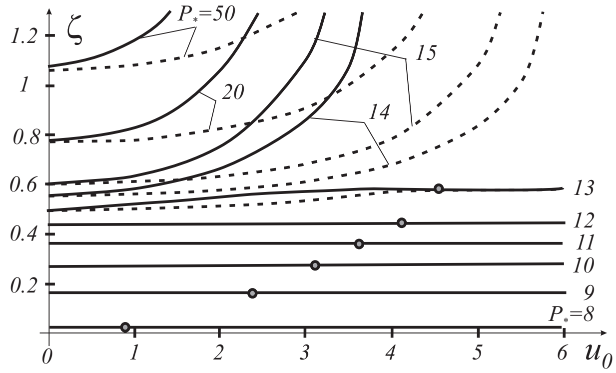


Fig. 12 Attraction basins for inextensible rod

with

$$F_{00}(u) = 0.02565 - 0.000221u^2 + 0.00000117u^4,$$

$$G_{00}(u) = -0.0406u + 0.00093u^3 - 0.00000736u^5.$$

Function $F(U)$ is odd. Condition $F'(0) > 0$ yields the boundary of stability of the vertical position under the support vibration $\zeta_* = 1.26(1 - 7.84/P_*)$ which is very close to Eq.(46).

According to Eq. (8) the initial conditions for Eq.(60) are as follows

$$U = u_0, \quad \frac{dU}{d\theta} = -u_1(u_0)\cos\beta \quad \text{at} \quad \theta = 0, \tag{62}$$

where β is the initial phase of vibration excitation.

For the inextensible rod the attraction basins are shown in Fig. 12 in plane $\{u_0, \zeta\}$ for various values of a weight-length parameter P_*. The resistance coefficient $n = 0.1$ is taken. For $P_* \leq 12$ the boundaries $\zeta_*(u_0)$ are approximately constant and do not depend on β. As a result, the absolute (G_a) and the partial (G_p) attraction basins coincide. For these values of P_* the equilibrium points u_* (see Table 2) marked by bold dots in Fig. 12 lie within the attraction basins. The case $P_* = 13$ is intermediate. For $P_* \geq 14$ the points u_* lie outside the attraction basins, and areas G_a and G_p differ from each other. The boundaries g^- of areas $(G_a : \ \zeta \geq \zeta_*(u_0))$ are pictured by continuous lines, and the boundaries g^+ of areas G_p are denoted by dashed lines.

For $P_* \geq 14$ the attraction basins lie at $u_0 < 6$, therefore the single mode approximation (50) is acceptable for their approximate construction.

For $P_* \leq 13$ the rod takes a curvilinear equilibrium position $u = u_*$ (see Fig. 10 with $\lambda = P_*$). For vibrations with $\zeta > \zeta_*$ the rod takes the vertical position. For $P_* \geq 14$, the initial condition $u_0 = u_*$ and under vibration the rod comes to another (non-vertical) position: $U(\theta) \to u_\infty$ at $\theta \to \infty$ with $F(u_\infty) = 0$. The stable vertical position can be achieved if the initial position u_0 of rod lies within the attraction basin (see the example below).

We consider, for example, the case $P_* = 15$, $\zeta = 0.7$. Then we have $u_* = 5.133$, $u_\infty = 3.157$, $u^- = 1.654$, $u^+ = 3.301$ where the points u^- and u^+ lie on the boundaries g^- and g^+, respectively. At $\theta \to \infty$ the following limit relations are valid:

$$\begin{aligned} \beta = 0: \qquad & U(\theta) \to 0 \text{ at } u_0 < u^-, \qquad U(\theta) \to u_\infty \text{ at } u_0 > u^-, \\ \beta = \pi: \qquad & U(\theta) \to 0 \text{ at } u_0 < u^+, \qquad U(\theta) \to u_\infty \text{ at } u_0 > u^+. \end{aligned}$$

3.5 The influence of longitudinal waves on stability of the vertical position and attraction basins of the extensible rod

For study we use Eq.(60) in which functions $u_1(U)$ and $v_1(U)$ satisfy the linear system (58).

From Eq.(60) we obtain the condition for stability of the vertical position $F'(0) > 0$ or

$$F'(0) = -(1/2)F'_{10}u'_1 v_1 - (1/4)F''_{20}v_1^2 - (1/2)G'_{00}u'_1 + \zeta^{-1}\left(\frac{c_0}{P_*} + G'_{00}\right) > 0, \quad (63)$$

where all functions are to be calculated at $U = 0$,

$$\begin{aligned} & v_1(0) = \frac{(5/2)\nu^2}{(5/2)-\nu^2}, \qquad u'_1(0) = -\frac{F'_{10}(0)v_1(0)+G'_{00}(0)}{F_{00}(0)}, \\ & \nu^2 = f\zeta, \quad f = \frac{Lg}{c^2\varepsilon^2} = \frac{L^3 g}{c^2 a^2}, \\ & F_{00}(0) = 0.02565,\ F'_{10}(0) = -0.005847,\ F''_{20}(0) = -0.00408,\ G'_{00}(0) = -0.04062. \end{aligned} \quad (64)$$

The single mode approximation (50) $e(s,t) = (1-s)v(t)$ is not sufficient for the complete analysis of extensible rod because higher longitudinal resonances (see Fig. 8) are not taken into account. We consider only the cases with $\nu \le \pi$. For the fixed values of parameter f the inequalities

$$\zeta \le \frac{\pi^2}{f} \qquad \text{or} \qquad \omega \le \frac{\pi c}{L}. \quad (65)$$

are to be fulfilled.

Fig. 13 displays the area of the vertical stability (area S between lines b_0 and b_1) under the support vibration for two values $P_* = 9$ and $P_* = 20$ of parameter P_*. The value $f = 0$ corresponds to the inextensible rod. The lower boundary decreases with growth of f. The lines c_1 and c_2 correspond to the first longitudinal resonance $\nu = \pi/2$ and curve $\nu = \pi$, respectively. The instability zone (U_1) is above line c_1. As for the domain above c_2, the results of the performed analysis are unreliable because the influence of second resonance is to be taken into account, see Fig. 8.

To construct the attraction basin of the vertical position we solve numerically the average equation (60), in which functions $u_1(U)$ and $v_1(U)$ are found from Eq.(58)

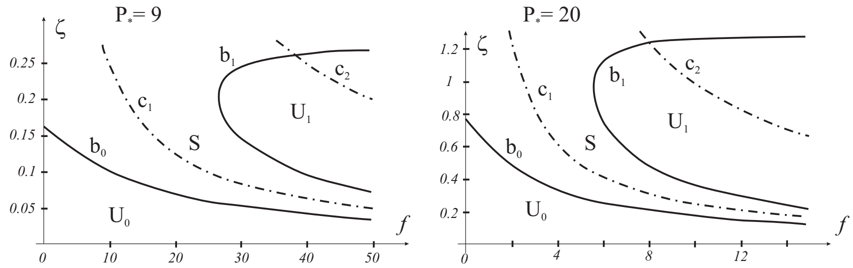


Fig. 13 Areas of the vertical position stability

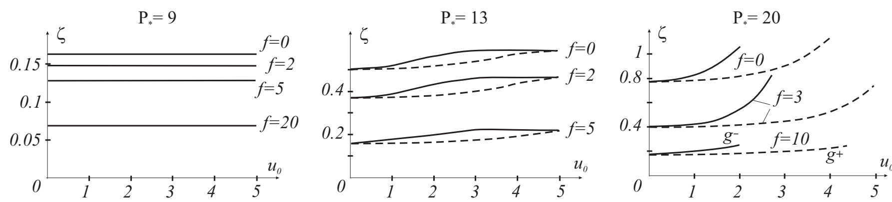


Fig. 14 Attraction basins for an extensible rod

with the initial conditions $U(0) = u_0,\ U_{,\theta}(0) = -u_1(u_0)\cos\beta$. For three values of parameter P_* the results are shown in Fig. 14. The results for the extensible rod basically repeat those for the inextensible rod (see Fig. 12), however there appears an additional parameter f describing extension. In all studied cases the boundaries g^- and g^+ decrease with growth of f. For small values of P_* (see $P_* = 9$ in Fig. 14) in the studied interval $0 \le u_0 \le 5$ the boundaries g^- and g^+ coincide and do not depend on u_0. In the intermediate case $P_* = 13$ the influence of initial phase β is essential and $g^- \le g^+$.

In the case $P_* = 20$ the attraction basins occupy areas smaller than $u_0 \le 5$ because Eq.(60) has a singular point at resonance (at $\zeta = 2.5/f$). As a result, three various kinds of behavior of solution of Eq.(60) are possible depending upon the parameters $(P_*, f, \zeta, u_0, \beta)$:

(i) $U(\theta) \to 0$ at $\theta \to \infty$, then the point lies in the attraction basin;

(ii) $U(\theta) \to u_* \ne 0$ at $\theta \to \infty$ with $F(u_*) = 0,\ F'(u_*) > 0$, then the point comes to the equilibrium state $u = u_*$;

(iii) $U(\theta) \to \infty$ at $\theta \to \theta_* < \infty$ then the point approaches the singular point of Eq.(60).

4 Discussion

Attraction basins of the vertical position for the Kapitsa's pendulum under action of various types of support motion are constructed. A two-scale asymptotic expansion is used, and the average motion of pendulum is shown to depend on a slow time. A peculiarity of the problem is that the average motion of pendulum is sensitive to the initial (positive or negative at the initial time instant) impulse that depends on the initial phase of excitation. As a result, the attraction basin consists of two areas: in the so-called absolute area G_a the pendulum comes to vertical position for any initial phase, whereas in the so-called partial area G_p it comes to the vertical position only for a part of initial phases. The averaged system of equations by Blekhman [4, 5] obtained as a result of introduction of the vibrational force correctly determines the stability conditions, however, it does not allow one to construct the attraction basin because it does not account for the initial pulse generated by the initial phase of the disturbance.

Motion of the Kapitsa's pendulum at random stationary vibration of support is basically similar to poly-harmonic vibrations with the following differences. The attraction basins obtained are not definite, and they can be described with some probability. Three kinds of attraction basins are constructed:

(i) theoretical attraction basins. The study is based solely upon the properties of spectral density of excitation;

(ii) attraction basins obtained by approximation of the random process by a sum of periodic terms with random amplitudes and phases. In the limit, (ii) tends to (i) in the mean-square sense with an unbounded increase in the number of terms. With a finite (albeit large) number of terms, the results differ markedly which is also mentioned in the present article;

(iii) attraction basins obtained by a probabilistic elaboration of numerical solution of equations with this sum as excitation. The study is based on the numerical simulation of random variables included in the sum and the subsequent numerical solution of equation (15). The behavior of the solution with increasing time is revealed. This numerical simulation is repeated many times and the results are processed by methods of mathematical statistics. Particularly, in the present paper the results of ten simulations are processed to construct the attraction basin boundary.

Motion of a flexible rod with a clamped lower end and a free upper end subjected to harmonic vertical vibration of support is described by a system of non-linear integro-differential equations in partial derivatives. The exact solution is not constructed. We suggested an approximate model with two degrees of freedom which is acceptable if the rod inclination from the vertical position is not considerable and the excitation frequency is smaller than the second natural frequency of longitudinal vibrations of rod. The two-scale expansions are used to this approximate model, too.

For the inextensible rod the dimensionless equations contain a universal parameter P_* that depends on weight, length and bending stiffness of the rod. For $P_* < 7.84$ the vertical position of rod is stable (without the support vibration) thus the cases of $P_* > 7.84$ are of special interest. For the extensible rod an additional wave parameter f turns out to be important as it describes the influence of longitudinal vibration of the rod.

First we considered an inextensible rod. It is established that for $P_* \leq 13$ the rod with some initial static curvilinear equilibrium state (see Fig. 10) achieves the vertical position under the support vibration of high intensity ζ. In these cases the static equilibrium position lies within the attraction basin. At $P_* \geq 14$ the equilibrium lies outside the attraction basin, and the rod with the same initial conditions comes under vibration to another equilibrium of average motion. To obtain a vertical limit position it is necessary to shift the initial position of rod within the attraction basin more close to the vertical, see Fig. 12.

We studied the impact of longitudinal vibration of the extensible rod on its stability. The area of stable vertical position has the upper boundary that lies upper the first resonance of longitudinal vibrations of rod, cf. see Fig. 13. As for an inextensible rod, the lower boundaries g^- and g^+ of the attraction basins at $P_* \leq 13$ in the studied interval $0 \leq u_0 \leq 5$ weakly depend on u_0 and essentially depend on the wave parameter f (see Fig. 14). For both inextensible (Fig. 12) and extensible (Fig. 14) rods, at small weight parameter $P_* \leq 12$ the attraction basins occupy all studied interval of initial position $u_0 \leq 6$, while for $P_* \geq 14$ the right boundaries of attraction basins in plane (u_0, ζ) move left, close to the vertical position.

The level ζ of support vibrations bringing the rod to vertical position for the extensible rod is lower than that for the inextensible rod.

In order to demonstrate how broad can be the attraction basins we refer to Ref. [5] in which the theoretical analysis carried out in the present paper got an experimental confirmation. Book [5] reported the following experiment performed by V.B.Vasil'kov. A soft rope of ca. 10 cm length and 1 cm diameter is clamped at the lower end while the upper end is free. The rope has such a low bending rigidity that the upper end lies on the support. Under intensive vertical vibration of the support in some frequency band the rope takes stable upward position regardless of the initial shape. The theoretical study presented here explains existence of stable vertical position, however it is not capable to describe the rope motion from the initial state to the stable vertical position as our analysis is restricted to the case of small inclinations ($u \leq 6$) of the rope from the vertical.

4.1 Conclusions

We consistently considered the attraction basins of the upward vertical position of the pendulum which is known to be unstable without support vibration. The previously obtained areas of the attraction basins for the pendulum upward position are given for harmonic, polyharmonic and random vibration of the support. The attraction basins for a vertical flexible inextensible and extensible rod with a free upper end are constructed. The analytical solutions are constructed in the classical Kapitsa problem whereas for a flexible rod one has to restrict oneself to an approximate solution of systems with one or two degrees of freedom. The method of two-scale expansions is used. In all cases, the attraction basin consists of two parts: absolute and partial. In the first of them, the attraction takes place for all initial phases of the perturbation, while in the second one, only for some initial phases. For the sake of consistency, our previously published results are given, and the necessary references are made to them.

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