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Determination of the Parameters of a Nonlinear Kinematic Galactic Rotation Model Based on the Proper Motions and Radial Velocities of Stars from the Gaia DR3 Catalogue

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Abstract—We have solved the Ogorodnikov–Milne stellar-kinematics equations in the Galactic rectangular coordinate system based on the total velocities for a special sample of stars with radial velocities from the final Gaia Data Release 3 catalogue. We have found the region of applicability of the linear model and the regions that it describes poorly. We have constructed a second-order model that takes into account the peculiarities of stellar kinematics more accurately and showed its applicability for stars at distances up to 5 kpc.

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INTRODUCTION

The construction of stellar-kinematics models for the stellar velocity field in circumsolar space has a long history. The proper motions of stars were first detected by E. Halley in 1718. It was suggested that the systematic effects in the proper motions of stars were caused by the motion of the Sun in space. Indeed, for the nearest stars this is the most significant effect. The solar apex was first determined by W. Herschel back in 1806. In 1859 M. Kowalski and, subsequently, G. Airy proposed an up-to-date method of determining the apex coordinates and derived equations that now bear their names.

The next most significant effect, Galactic rotation, began to be actively studied in the 1920s. In 1925–1927 Lindblad (1927) assumed that the rotating system consisted of a number of subsystems rotating around a single axis, but with different velocities. In 1927 Oort (1927) tested the hypothesis of Galactic rotation observationally. He derived formulas to determine the rotation both from the proper motions of stars and from their radial velocities. In these formulas no assumptions about the character of rotation ("rigid-body," "Keplerian") are made. The angular velocity of Galactic rotation $\Omega(R)$ is represented as a segment of the Taylor series limited by the first two terms. Generally, these equations are a special case of Bottlinger's formulas, which are described in detail in Bobylev (2007) and Bobylev et al. (2014).

Before the introduction of the J2000.0 system, parameters to the precession constant were also included into the stellar-kinematics equations (Fricke 1977; Vityazev 2000).

In 1930 Ogorodnikov (1958) outlined a more general approach to analyzing the proper motions of stars based on Helmholtz's theorem for the motion of continuous media. A classic form of these equations, often called the Ogorodnikov–Milne model, can be found, for example, in Clube (1972).

In the second half of the 20th century the attention of researchers was drawn to the anomalies of local kinematics, often called the "local system of stars" (Shatsova 1950). This was explained primarily by the observational material, since the catalogues at that time contained mostly nearby stars, while stars at distances closer than 100 pc indeed have special kinematics (Tsvetkov 1995). The Hipparcos catalogue (Perryman 1997) allowed the kinematics of nearby stars to be investigated quite accurately. However, despite the revolutionary significance of this catalogue, it contained mostly nearby stars. The presence of individual parallaxes allowed the parameters of the local system of stars to be estimated reliably (Tsvetkov 1998).

With the appearance of extensive stellar catalogues, such as XPM (Fedorov et al. 2009), PPMXL (Roeser et al. 2010), and UCAC4 (Zacharias et al.

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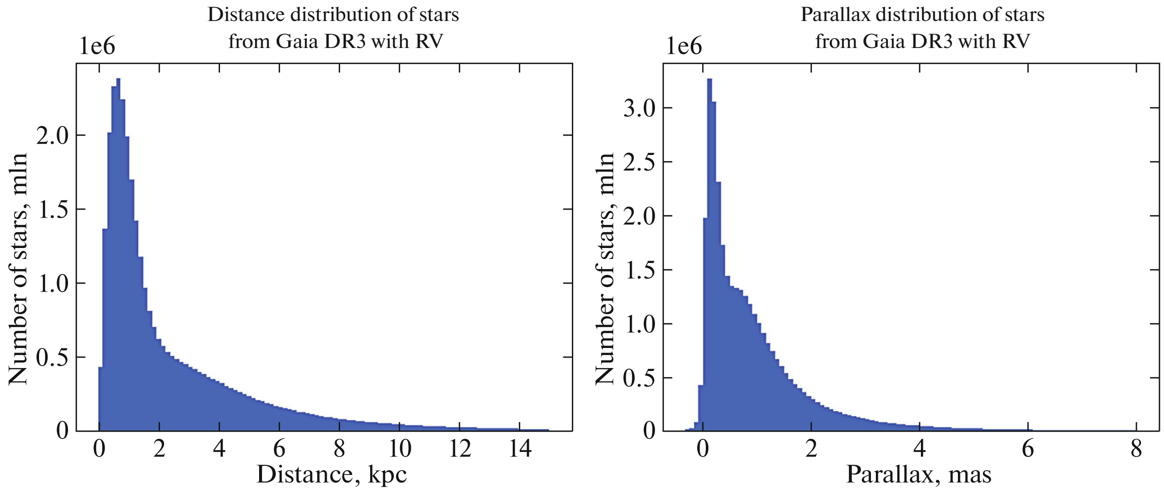

Fig. 1. Distributions of stars from Gaia DR3 with RV in distance and parallax.

2013), it became possible to investigate the kinematics of large volumes of the Galaxy. However, the absence of individual parallaxes allowed the statistical distances to stars to be determined only by indirect methods. Nevertheless, new formal kinematic models using scalar and vector spherical harmonics were based on the material of these (and earlier) catalogues. These studies confidently showed the presence of systematic components in the proper motions of stars that are not described by the complete linear Ogorodnikov–Milne model (Vityazev and Tsvetkov 2013).

Searching for new models led to an extension of the linear model by additional components, which became possible with the publication of the latest releases of the Gaia catalogue (ESA, Gaia), since almost any nonlinear models require knowing the distances. The main approach used in these models involves an expansion of the angular velocity of Galactic rotation into a higher-order Taylor series. Of special note is the paper by Bobylev and Bajkova (2023), where the angular velocity was expanded to the fourth order and the expansion coefficients were determined.

In this paper we construct a complete second-order model including the derivatives of all parameters of the linear Ogorodnikov–Milne kinematic model in three directions. As observational material we use the entire subset of Gaia stars with radial velocities.

CHARACTERISTICS OF THE GAIA DR3 WITH RV CATALOGUE

The Gaia DR3 catalogue contains information about 1.8 billion stars in our Galaxy (ESA, Gaia); 1.47 billion stars have data on both proper motions and parallaxes. To construct a complete three-dimensional model of stellar velocities, it is necessary to also have information about the radial velocities. The Gaia DR3 catalogue contains 33 812 183 stars with information about their radial velocities, giving us all three components of the stellar space velocity. This subcatalogue may be called "Gaia DR3 with RV". This is the most important supplement to the previous version of the Gaia Early Release 3 catalogue (ESA, Gaia EDR3), from which the astrometric part (namely the coordinates, parallaxes, and proper motions) entered into the final third version (Brown et al. 2021) almost without changes. The next version is expected no earlier than 2025.

Most of the stars in the Gaia DR3 with RV subcatalogue are concentrated at distances from 0 to 2 kpc, but there are also stars at distances greater than 10 kpc (Fig. 1). Despite the fact that the authors of Gaia claim the accuracy of parallaxes to be 0.01 milliarcseconds (mas) for stars brighter than 15^m, the actual accuracy of Gaia (at least of the current version) has turned out to be considerably lower. In the full catalogue more than 15% of the stars have a negative parallax (Tsvetkov 2021). At the same root-mean-square error and different parallaxes, it is natural to expect that the relative distance estimation accuracy will be lower for more distant stars. In particular, the relative accuracy of the parallaxes of stars located at distances of a few kpc can be even poorer than 100%. However, in our subset most of the stars are nearby and have a relative parallax accuracy better than 1%, which is evidence for a higher astrometric quality of the Gaia DR3 with RV catalogue.

An interesting feature can be noticed in the spatial distribution of stars. Figure 2 presents the density of the distribution of stars in a thin disk of thickness 200 pc in the Galactic XY plane. Dark "rays" are clearly seen. We interpret them as possible manifestations of dust that shields distant regions and, for

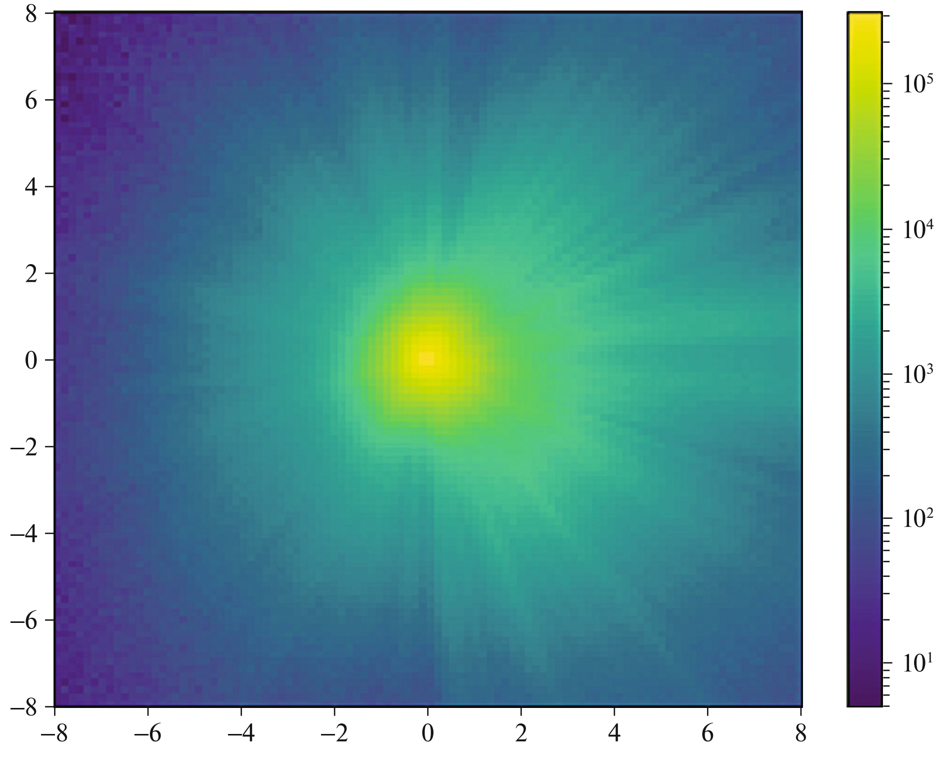

Fig. 2. Density of the distribution of stars from Gaia DR3 with RV in projection onto the Galactic plane. The units of measurement along the axes are kpc, the Galactic center is on the right.

this reason, we record a smaller number of stars in them. However, this fact needs a further study.

It should be noted that the stars with radial velocities probably also have a better, in general, astrometric accuracy. In this sample the parallax accuracy is better than 1 mas for 99.5% of the stars and the relative parallax accuracy is better than 10% for 76.7% (Fig. 3), whereas in the full catalogue only 5% have a relative parallax accuracy better than 10%.

The proper motions have a highly satisfactory accuracy (Fig. 4). The relative proper motion accuracy is better than 20% almost for all of the stars in our list.

As regards the radial velocities, the initial Gaia plan was greatly reduced, and one should not expect an increase in the number of stars with information about their radial velocities. The accuracy of the stellar radial velocities in the Gaia DR3 with RV subcatalogue themselves is low, being $\sim$3 km s^{-1}. This leads to a fairly high relative radial velocity error (Fig. 5).

Nevertheless, there is a unique material at our disposal: 30 million stars with all three spatial coordinates and three velocity components. This allows one to pass to a rectangular coordinate system in which many of the kinematic effects are seen more clearly. To carry out our calculations, we restricted ourselves to stars with distances up to 8 kpc (30 667 161 of the 33 812 183 stars in Gaia DR3 with RV). The stars with negative parallaxes were discarded.

We deliberately make no additional restrictions on the sample (by color index, luminosity, and other possible characteristics), since our work is rather mathematical, proposing a new method of describing the stellar kinematics of a large sample of stars.

In our calculations we used simple estimates of the distance $1/\pi$, although other methods are also used (see, e.g., Bailer-Jones et al. 2021).

THE OGORODNIKOV–MILNE MODEL IN CARTESIAN COORDINATES

In view of the peculiarities of constructing any stellar catalogues and, particularly, our poor knowledge of parallaxes, the representation of stellar proper motion and radial velocity models in a spherical (equatorial or Galactic) heliocentric coordinate system is traditionally used. The derivation of such equations is presented in detail, for example, in Tsvetkov and Amosov (2019). This approach spawns

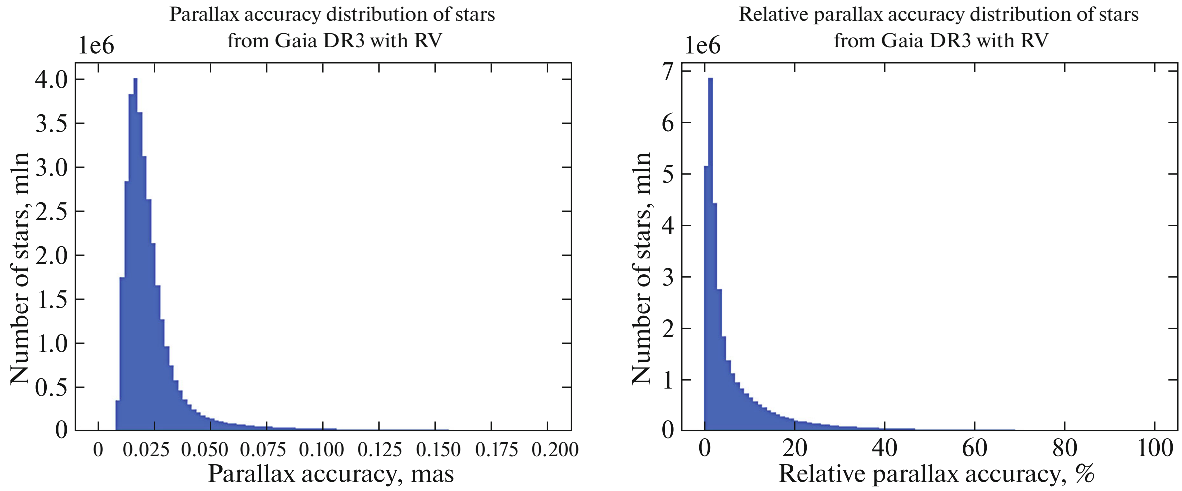

Fig. 3. Distributions of stars from Gaia DR3 with RV in absolute and relative parallax accuracies.

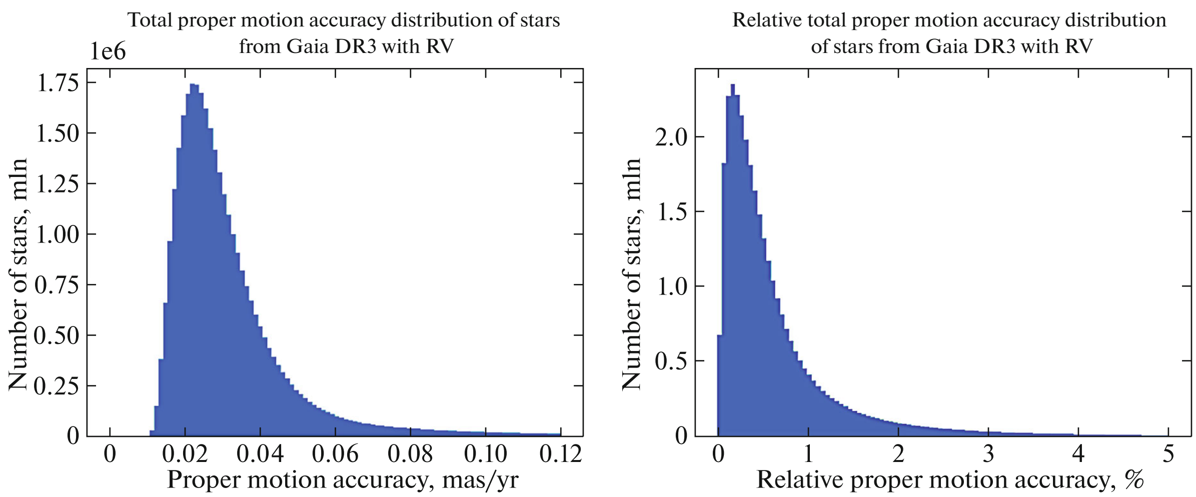

Fig. 4. Distributions of stars from Gaia DR3 with RV in absolute and relative accuracies of their total proper motions.

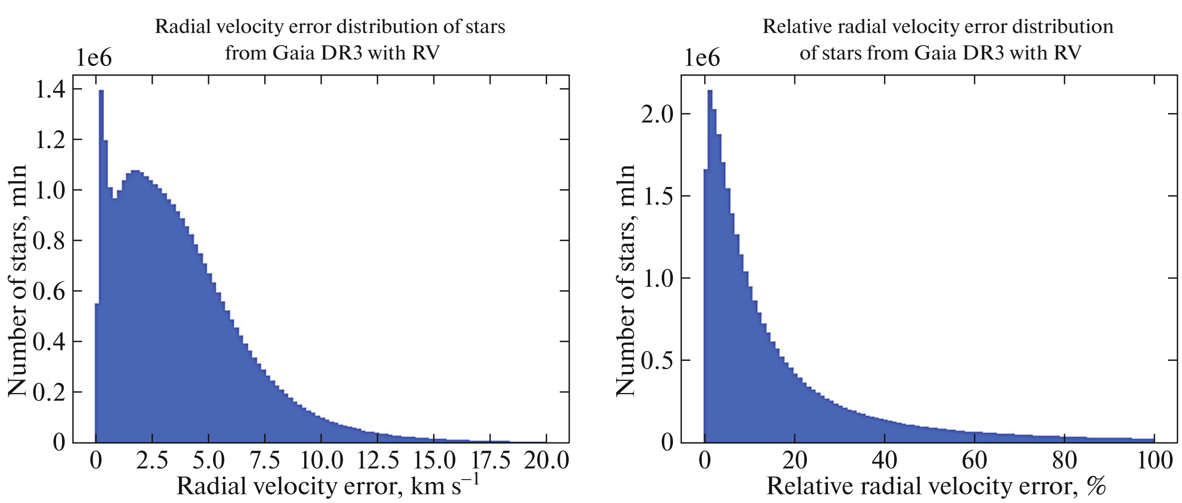

Fig. 5. Distributions of stars from Gaia DR3 with RV in absolute and relative accuracies of their radial velocities.

rather cumbersome equations with an abundance of trigonometric functions, but it has remained the only justified one for a long time. In contrast, having three spatial coordinates and all three velocity components, the total velocity of a star can be represented in the Galactic rectangular coordinate system in a very simple form (Ogorodnikov 1965):

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} -U \\ -V \\ -W \end{pmatrix} + \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{1}$$
$$\times \begin{pmatrix} x \\ y \\ x \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix},$$

where U, V, and W are the velocity of the Sun relative to the local standard of rest, ω_x, ω_y, and ω_z are the rigid-body rotation parameters of the stellar system, $\mathbf{M}$ is the symmetric deformation tensor of the velocity field, and x, y, and z are the Galactic Cartesian stellar coordinates.

Representation (1) is transformed to the form

$$\begin{aligned} v_x &= -U + M_{11}x + (M_{12} - \omega_z)y \\ &\quad + (M_{13} + \omega_y)z, \\ v_y &= -V + (M_{12} + \omega_z)x + M_{22}y + (M_{23} - \omega_x)z, \\ v_z &= -W + (M_{13} - \omega_y)x + (M_{23} + \omega_x)y + M_{33}z. \end{aligned} \tag{2}$$

If we redesignate the coefficients at x, y, and z, then we will obtain a linear decomposition of the velocity in the Cartesian coordinate system:

$$\begin{aligned} v_x &= -U + a_x x + a_y y + a_z z, \\ v_y &= -V + b_x x + b_y y + b_z z, \\ v_z &= -W + c_x x + c_y y + c_z z. \end{aligned} \tag{3}$$

The coefficients of this system can be easily determined by the least-squares method. All of the Ogorodnikov–Milne model parameters (2) are unambiguously derived from them through elementary transformations.

The result of our calculation of the linear model parameters from 30 million nearest stars in the Gaia DR3 with RV subcatalogue is presented in Table 1. The parameters of the Ogorodnikov–Milne model given in Table 2 correspond to this solution.

THE REGION OF APPLICABILITY OF THE LINEAR MODEL

The region of applicability of the linear kinematic model is thought to be no more than 1–1.5 kpc from the Sun, with local anomalies of stellar kinematics, as has already been said previously, manifesting themselves in the immediate solar neighborhood. In our previous paper (Tsvetkov 2022) we showed the stability of the Ogorodnikov–Milne model parameters for samples up to 1 kpc and even, unexpectedly for us, for samples at great distances with one exception—the parameter of the solar motion V along the Y Galactic rotation axis begins to increase with distance. We will reveal the cause of this phenomenon. In this paper we perform the solution for the entire group of stars (since we take into account the individual stellar parallaxes), but here we also see an anomalously higher velocity of the Sun V (Table 2). In the same paper (Tsvetkov 2022) we demonstrated that distant stars make a major contribution to the increase in this parameter.

We can make sure that the linear model is inapplicable farther than 1 kpc by examining the behavior of the residual velocities with increasing distance. The residual velocity is the velocity of a star minus the velocity calculated from the model

$$\begin{aligned} dv_x &= v_x - (-U + a_x x + a_y y + a_z z), \\ dv_y &= v_y - (-V + b_x x + b_y y + b_z z), \\ dv_z &= v_z - (-W + c_x x + c_y y + c_z z). \end{aligned} \tag{4}$$

The derived residual velocities dv_x, dv_y, and dv_z can be depicted in projection onto the Galactic XY plane (Fig. 6). The numerical values of the residual velocities along the X and Y axes are presented in Tables 3 and 4.

We see that the residual stellar velocities are less than 10 km s^{-1} in a small solar neighborhood. At greater distances the difference between them begins to increase and reaches tens of km s^{-1}. In Fig. 6 the region of low residual velocities is clearly seen and is elongated along the direction of Galactic rotation.

THE QUADRATIC MODEL

A natural extension of the linear model (4) is the quadratic model (5). In this case, the coefficients at the second-order functions are linear combinations of the partial derivatives of various Ogorodnikov–Milne model parameters in different directions:

$$\begin{aligned} v_x &= -U + a_x x + a_y y + a_z z + a_{xx}x^2 \\ &\quad + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{xz}xz + a_{yz}yz, \\ v_y &= -V + b_x x + b_y y + b_z z + b_{xx}x^2 \\ &\quad + b_{yy}y^2 + b_{zz}z^2 + b_{xy}xy + b_{xz}xz + b_{yz}yz, \\ v_z &= -W + c_x x + c_y y + c_z z + c_{xx}x^2 \\ &\quad + c_{yy}y^2 + c_{zz}z^2 + c_{xy}xy + c_{xz}xz + c_{yz}yz. \end{aligned} \tag{5}$$

Table 1. The coefficients of the linear model determined from Gaia DR3 with RV stars within 8 kpc

U	11.264 ± 0.009	a_x	-0.865 ± 0.005	a_y	27.823 ± 0.005	a_z	-0.343 ± 0.015
V	30.753 ± 0.009	b_x	-11.174 ± 0.004	b_y	0.338 ± 0.005	b_z	-0.576 ± 0.014
W	7.906 ± 0.006	c_x	-0.437 ± 0.003	c_y	0.217 ± 0.003	c_z	0.212 ± 0.009
	km s^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}

Table 2. The parameters of the Ogorodnikov–Milne model determined from Gaia DR3 with RV stars within 8 kpc

U	11.264 ± 0.009	ω_x	0.397 ± 0.007	M_{11}	-0.865 ± 0.005	M_{12}	8.324 ± 0.003
V	30.753 ± 0.009	ω_y	0.047 ± 0.008	M_{22}	0.338 ± 0.005	M_{13}	-0.390 ± 0.008
W	7.906 ± 0.006	ω_z	-19.498 ± 0.003	M_{33}	0.212 ± 0.009	M_{23}	-0.180 ± 0.007
	km s^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}

Table 3. Averaged values of the residual velocities dv_x, dv_y, and dv_z for the samples along the X axis of the Galactic coordinate system for stars with $|y| < 100$ pc and $|z| < 100$ pc. The distance to the left boundary of the sample in kpc is specified as x. The residual velocities are given in km s^{-1}

x	−5.0	−4.5	−4.0	−3.5	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
dv_x	−9.6	−9.4	−8.5	−6.6	−5.8	−4.2	0.4	2.8	5.4	1.8	0.5	−3.0	−3.0	0.4	1.3	1.6	1.1	2.4	7.4	6.8
dv_y	−53.0	−45.9	−40.3	−34.8	−28.9	−21.0	−15.5	−8.3	0.2	6.8	11.8	19.3	25.0	29.5	29.9	31.5	31.8	29.9	22.6	26.3
dv_z	2.2	0.4	0.0	0.1	0.0	0.2	0.3	−0.5	0.2	0.0	0.4	0.8	0.6	0.5	0.4	−0.5	−0.7	−1.2	−0.2	−1.4

Table 4. Averaged values of the residual velocities dv_x, dv_y, and dv_z for the samples along the Y axis of the Galactic coordinate system for stars with $|x| < 100$ pc and $|z| < 100$ pc. The distance to the left boundary of the sample in kpc is specified as y. The residual velocities are given in km s^{-1}

y	−5.0	−4.5	−4.0	−3.5	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
dv_x	17.2	7.1	2.2	8.3	3.0	2.4	1.7	0.6	0.4	1.3	1.5	1.7	1.6	−0.2	−0.6	−3.2	−3.6	−6.9	−8.9	−7.5
dv_y	−9.5	−5.6	−8.9	−1.6	−1.4	0.5	4.1	8.6	9.8	8.9	9.5	10.6	8.9	5.2	0.9	−3.5	−8.9	−12.6	−16.6	−21.7
dv_z	−3.3	−0.8	−1.3	0.2	−0.1	−0.1	0.2	0.6	0.5	0.3	0.2	0.5	0.5	0.3	0.2	−0.6	−0.7	−1.4	−1.4	−1.2

The result of our calculation of the coefficients in Eq. (5) by the least-squares method from 30 million Gaia DR3 with RV stars is presented in Table 5.

We see that $V = 21.64$ km s^{-1} took its usual value obtained for the immediate solar neighborhood. Thus, this model, on the one hand, provides reliable parameters of the solar motion relative to the local standard of rest and, on the other hand, works at great distances, up to the Galactic center, due to the quadratic terms.

By analogy with the residual velocities of the linear model dv_x, dv_y, and dv_z, we can calculate the residual velocities of the quadratic model dv'_x, dv'_y, and dv'_z.

Table 5. Coefficients of the quadratic model constructed from Gaia DR3 with RV stars

U	11.405 ± 0.010	a_x	-0.493 ± 0.006	a_y	25.046 ± 0.005	a_z	-0.404 ± 0.017
V	21.644 ± 0.008	b_x	-4.774 ± 0.005	b_y	-0.023 ± 0.005	b_z	-1.83 ± 0.014
W	7.838 ± 0.006	c_x	-0.507 ± 0.004	c_y	0.190 ± 0.003	c_z	0.148 ± 0.011
	km s^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}

a_{xx}	-0.507 ± 0.004	a_{yy}	0.049 ± 0.001	a_{zz}	0.065 ± 0.007
b_{xx}	-1.811 ± 0.001	b_{yy}	-1.115 ± 0.001	b_{zz}	-8.727 ± 0.006
c_{xx}	0.037 ± 0.001	c_{yy}	-0.036 ± 0.001	c_{zz}	-0.181 ± 0.004
a_{xy}	2.306 ± 0.002	a_{xz}	0.092 ± 0.005	a_{yz}	-0.122 ± 0.006
b_{xy}	0.117 ± 0.002	b_{xz}	0.273 ± 0.004	b_{yz}	0.332 ± 0.005
c_{xy}	0.027 ± 0.001	c_{xz}	0.060 ± 0.003	c_{yz}	0.312 ± 0.004
	km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}		km s^{-1} kpc^{-1}

Table 6. Averaged values of the residual velocities dv'_x, dv'_y, and dv'_z for the samples along the X axis of the Galactic coordinate system for stars with $|y| < 100$ pc and $|z| < 100$ pc. The distance to the left boundary of the sample in kpc is specified as x. The residual velocities are given in km s^{-1}

x	−5.0	−4.5	−4.0	−3.5	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
dv_x	−5.5	−5.9	−6.0	−5.0	−3.8	−1.9	0.9	3.3	4.4	2.5	0.0	−2.6	−2.6	−1.7	−1.8	0.4	2.4	3.4	5.5	5.2
dv_y	11.9	7.4	2.4	−1.6	−3.7	−5.2	−6.3	−5.9	−3.4	−0.8	2.7	6.6	8.5	9.7	11.5	14.4	16.3	17.1	17.7	18.6
dv_z	0.8	−0.4	−0.2	−0.5	−0.3	−0.1	−0.3	−0.6	−0.2	0.2	0.3	0.5	0.4	0.5	0.3	−0.1	−0.5	−0.8	−1.4	−2.5

Table 7. Averaged values of the residual velocities dv'_x, dv'_y, and dv'_z for the samples along the Y axis of the Galactic coordinate system for stars with $|x| < 100$ pc and $|z| < 100$ pc. The distance to the left boundary of the sample in kpc is specified as r. The residual velocities are given in km s^{-1}

r	−5.0	−4.5	−4.0	−3.5	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
dv_x	0.7	−1.9	−3.6	−4.9	−5.9	−5.9	−4.6	−3.3	−1.7	−0.1	1.8	3.0	4.3	5.5	6.3	6.0	5.0	4.0	2.3	−0.6
dv_y	−0.4	−1.6	−2.0	−1.4	−0.5	0.8	2.6	3.9	3.6	2.3	2.6	4.2	5.8	5.4	3.1	1.2	−0.1	−0.6	−0.7	−0.4
dv_z	0.2	−0.2	0.2	0.2	0.1	0.0	0.1	0.2	0.2	0.2	0.3	0.4	0.3	0.0	−0.2	−0.4	−0.2	−0.6	−0.7	−0.7

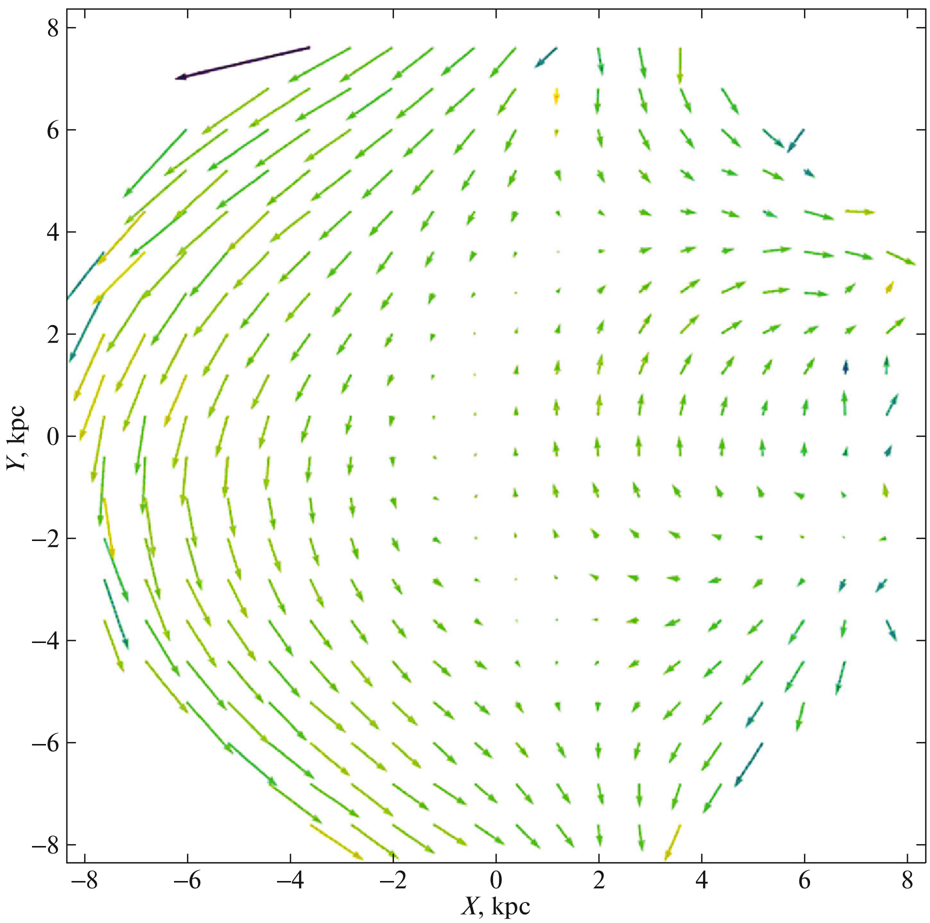

Fig. 6. Residual velocities dv_x, dv_y, and dv_z of stars with $|z| < 100$ kpc in projection onto the Galactic XY plane for stars within 8 kpc. The Galactic center is on the right. Each arrow corresponds to the projected averaged velocity in a column along the Z axis.

These data are given in Tables 6 and 7 and are illustrated in Fig. 7.

The data in Tables 6 and 7 show that the region of applicability of the quadratic model is considerably wider than that for the linear one and can reach 4–5 kpc, except for the small region closer to the Galactic center (Fig. 8). The presence of systematic components in the residual velocities of stars farther than 5 kpc can be due to both more complex Galactic rotation and an insufficient number of stars with a good accuracy of all six kinematic parameters at such distances.

THE MEANING OF THE QUADRATIC MODEL PARAMETERS

To explain the physical meaning of the coefficients at the quadratic functions, let us represent the coefficients of the linear model not as constants common to the entire sample, but as functions of x, y, and z:

$$
\begin{aligned}
v_x(x,y,z) &= -U + a_x(x,y,z)x \qquad (6)\\
&\quad + a_y(x,y,z)y + a_z(x,y,z)z,\\
v_y(x,y,z) &= -V + b_x(x,y,z)x\\
&\quad + b_y(x,y,z)y + b_z(x,y,z)z,\\
v_z(x,y,z) &= -W + c_x(x,y,z)x\\
&\quad + c_y(x,y,z)y + c_z(x,y,z)z,
\end{aligned}
$$

where

$$
\begin{aligned}
a_x(x,y,z) &= M_{11}(x,y,z), \qquad (7)\\
a_y(x,y,z) &= M_{12}(x,y,z) - \omega_z(x,y,z),\\
a_z(x,y,z) &= M_{13}(x,y,z) + \omega_y(x,y,z),\\
b_x(x,y,z) &= M_{12}(x,y,z) + \omega_z(x,y,z),\\
b_y(x,y,z) &= M_{22}(x,y,z),\\
b_z(x,y,z) &= M_{23}(x,y,z) - \omega_z(x,y,z),\\
c_x(x,y,z) &= M_{13}(x,y,z) - \omega_y(x,y,z),\\
c_y(x,y,z) &= M_{23}(x,y,z) + \omega_x(x,y,z),\\
c_z(x,y,z) &= M_{33}(x,y,z).
\end{aligned}
$$

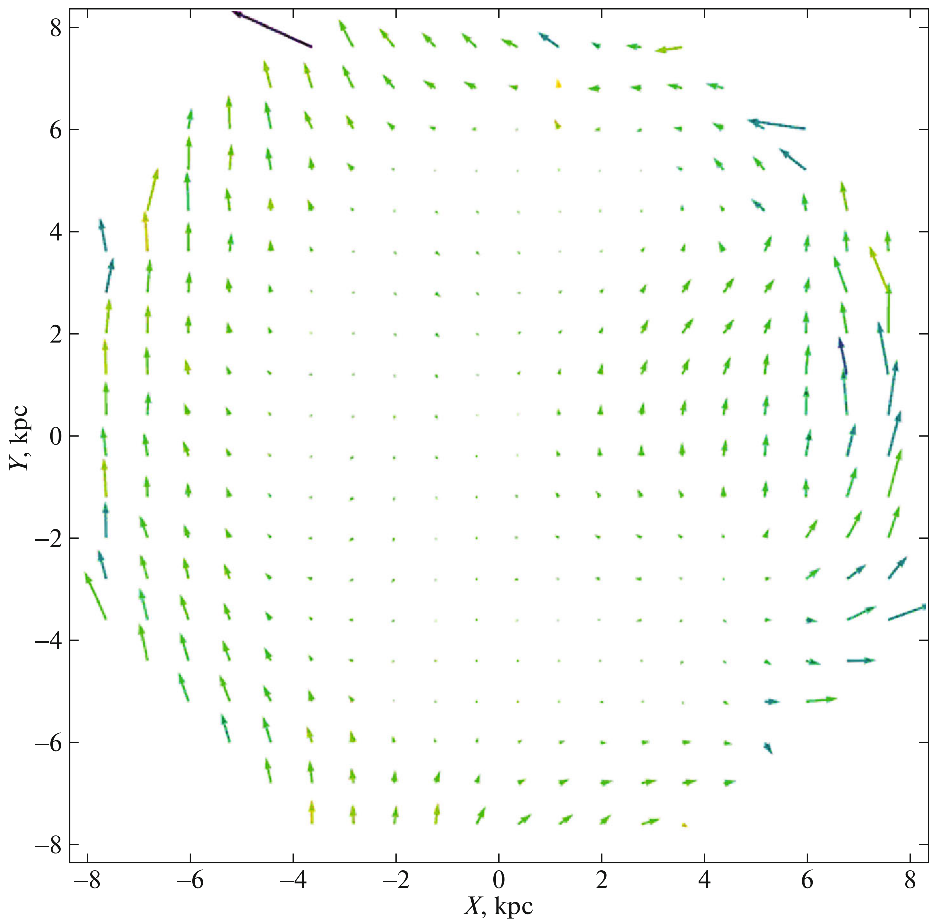

Fig. 7. Residual velocities dv'_x, dv'_y, and dv'_z of stars with $|z| < 100$ pc in projection onto the Galactic XY plane. The Galactic center is on the right. Each arrow corresponds to the projected averaged velocity in a column along the Z axis.

To estimate the possible values of these derivatives at zero, we will use a Taylor (Maclaurin) series expansion of the velocity field. Using the formula for the expansion of a function of three variables $f(x, y, z)$ into a Taylor series to the second order (8), it is easy to obtain this expansion for the functions v_x, v_y, and v_z near zero (9):

$$f(x,y,z) = f_0 + x\frac{\partial f_0}{\partial x} + y\frac{\partial f_0}{\partial y} + z\frac{\partial f_0}{\partial z} + \frac{x^2}{2}\frac{\partial^2 f_0}{\partial x^2} + \frac{y^2}{2}\frac{\partial^2 f_0}{\partial y^2} + \frac{z^2}{2}\frac{\partial^2 f_0}{\partial z^2} + xy\frac{\partial^2 f_0}{\partial x \partial y} + xz\frac{\partial^2 f_0}{\partial x \partial z} + yz\frac{\partial^2 f_0}{\partial y \partial z} + o(x^2 + y^2 + z^2), \tag{8}$$

$$v_x = -U + a_x x + a_y y + a_z z + \frac{\partial a_x}{\partial x}x^2 + \frac{\partial a_y}{\partial y}y^2 + \frac{\partial a_z}{\partial z}z^2 + \left(\frac{\partial a_x}{\partial y} + \frac{\partial a_y}{\partial z}\right)xy + \left(\frac{\partial a_x}{\partial z} + \frac{\partial a_z}{\partial x}\right)xz + \left(\frac{\partial a_y}{\partial z} + \frac{\partial a_z}{\partial y}\right)yz, \tag{9}$$

$$v_y = -V + b_x x + b_y y + b_z z + \frac{\partial b_x}{\partial x}x^2 + \frac{\partial b_y}{\partial y}y^2 + \frac{\partial b_z}{\partial z}z^2 + \left(\frac{\partial b_x}{\partial y} + \frac{\partial b_y}{\partial z}\right)xy + \left(\frac{\partial b_x}{\partial z} + \frac{\partial b_z}{\partial x}\right)xz + \left(\frac{\partial b_y}{\partial z} + \frac{\partial b_z}{\partial y}\right)yz,$$

$$v_z = -W + c_x x + c_y y + c_z z + \frac{\partial c_x}{\partial x}x^2 + \frac{\partial c_y}{\partial y}y^2 + \frac{\partial c_z}{\partial z}z^2 + \left(\frac{\partial c_x}{\partial y} + \frac{\partial c_y}{\partial z}\right)xy + \left(\frac{\partial c_x}{\partial z} + \frac{\partial c_z}{\partial x}\right)xz + \left(\frac{\partial c_y}{\partial z} + \frac{\partial c_z}{\partial y}\right)yz.$$

Here, for a more compact form, we keep in mind that the values of all coefficients and derivatives are taken at point $(0, 0, 0)$:

$$a_x = a_x(0,0,0), \quad a_y = a_z y(0,0,0), \quad \ldots;$$

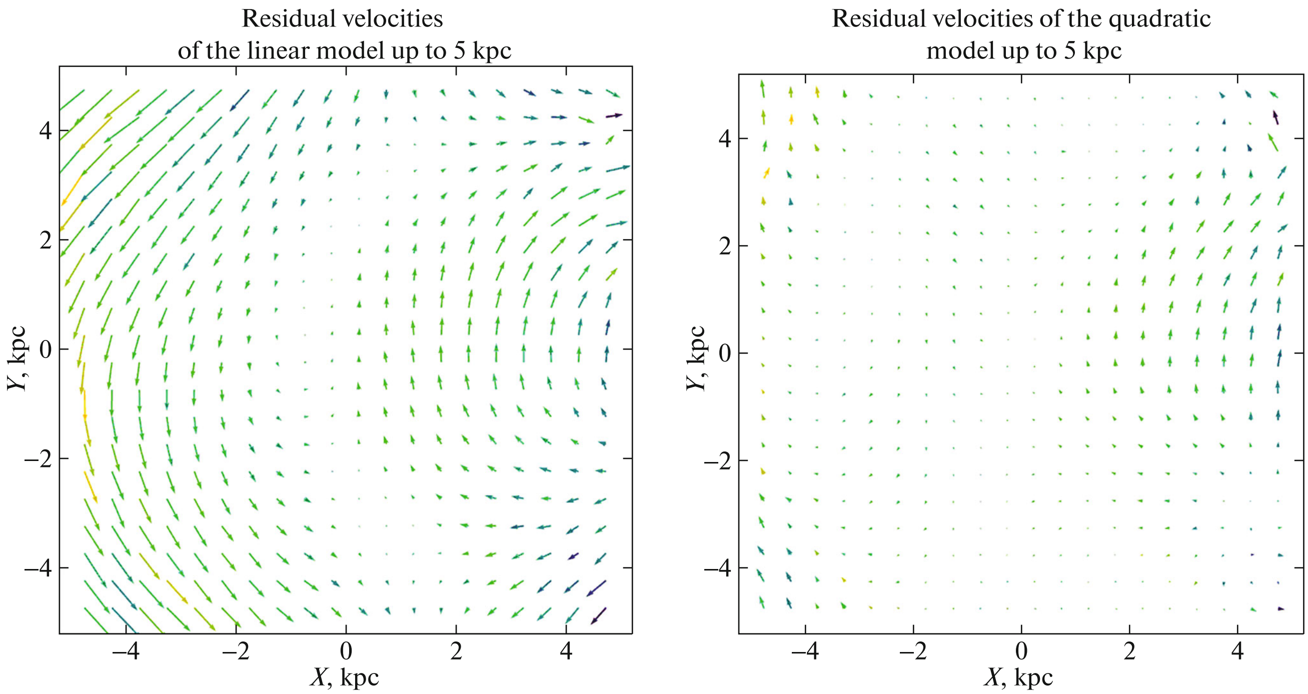

Fig. 8. Pictorial comparison of the residual velocities of the linear model dv_x, dv_y, dv_z (left) with the residual velocities of the quadratic model dv'_x, dv'_y, dv'_z (right) on a larger scale (± 5 kpc). The residual velocities are given in one scale.

$$\frac{\partial a_x}{\partial x} = \frac{\partial a_x}{\partial x}(0,0,0), \quad \frac{\partial a_y}{\partial y} = \frac{\partial a_y}{\partial y}(0,0,0), \quad \ldots .$$

Substituting (7) into (9) gives

$$\begin{aligned}
v_x = {} & -U + M_{11}x + (M_{12} - \omega_z)y \qquad (10) \\
& + (M_{13} + \omega_y)z + \frac{\partial M_{11}}{\partial x}x^2 \\
& + \left(\frac{\partial M_{12}}{\partial y} - \frac{\partial \omega_z}{\partial y}\right) y^2 + \left(\frac{\partial M_{13}}{\partial z} + \frac{\partial \omega_y}{\partial z}\right) z^2 \\
& + \left(\frac{\partial M_{11}}{\partial y} + \frac{\partial M_{12}}{\partial x} - \frac{\partial \omega_z}{\partial x}\right) xy \\
& + \left(\frac{\partial M_{11}}{\partial z} + \frac{\partial M_{13}}{\partial x} + \frac{\partial \omega_y}{\partial x}\right) xz \\
& + \left(\frac{\partial M_{13}}{\partial y} + \frac{\partial \omega_y}{\partial y} + \frac{\partial M_{12}}{\partial z} - \frac{\partial \omega_z}{\partial z}\right) yz, \\
v_y = {} & -V + (M_{12} + \omega_z)x + M_{22}y \\
& + (M_{23} - \omega_x)z + \left(\frac{\partial M_{12}}{\partial x} + \frac{\partial \omega_z}{\partial x}\right) x^2 \\
& + \frac{\partial M_{22}}{\partial y}y^2 + \left(\frac{\partial M_{23}}{\partial z} - \frac{\partial \omega_x}{\partial z}\right) z^2 \\
& + \left(\frac{\partial M_{22}}{\partial x} + \frac{\partial M_{12}}{\partial y} + \frac{\partial \omega_z}{\partial y}\right) xy \\
& + \left(\frac{\partial M_{23}}{\partial x} - \frac{\partial \omega_x}{\partial x} + \frac{\partial M_{12}}{\partial z} + \frac{\partial \omega_z}{\partial z}\right) xz \\
& + \left(\frac{\partial M_{23}}{\partial y} - \frac{\partial \omega_x}{\partial y} + \frac{\partial M_{22}}{\partial z}\right) yz, \\
v_z = {} & -W + (M_{13} - \omega_y)x + (M_{23} + \omega_x)y \\
& + M_{33}z + \left(\frac{\partial M_{13}}{\partial x} - \frac{\partial \omega_y}{\partial x}\right) x^2 \\
& + \left(\frac{\partial M_{23}}{\partial y} + \frac{\partial \omega_x}{\partial y}\right) y^2 + \frac{\partial M_{33}}{\partial z}z^2 \\
& + \left(\frac{\partial M_{23}}{\partial x} + \frac{\partial \omega_x}{\partial x} + \frac{\partial M_{13}}{\partial y} - \frac{\partial \omega_y}{\partial y}\right) xy \\
& + \left(\frac{\partial M_{33}}{\partial x} + \frac{\partial M_{13}}{\partial z} - \frac{\partial \omega_y}{\partial z}\right) xz \\
& + \left(\frac{\partial M_{33}}{\partial y} + \frac{\partial M_{23}}{\partial z} + \frac{\partial \omega_x}{\partial z}\right) yz.
\end{aligned}$$

Let us compare Table 5 and Eq. (10), write out the coefficients at the quadratic terms, and highlight the four most significant coefficients in boldface (Table 8).

Since there are only 18 quadratic terms, while there are 27 partial derivatives of the Ogorodnikov–Milne model parameters in them, it is impossible to determine all of them independently. We can obtain only some of their linear combinations. It should be said that when determining the parameters of the deformation tensor from the proper motions alone, we encounter such a problem when determining the parameters of the standard kinematic model as well.

Table 8. Linear combinations of the partial derivatives of the Ogorodnikov–Milne model parameters at zero point. The most significant coefficients are highlighted in boldface

Parameters	Formulas	Values
a_{xx}	$\frac{\partial M_{11}}{\partial x}$	-0.507 ± 0.004
a_{yy}	$\frac{\partial M_{12}}{\partial y} - \frac{\partial \omega_z}{\partial y}$	0.049 ± 0.001
a_{zz}	$\frac{\partial M_{13}}{\partial z} + \frac{\partial \omega_y}{\partial z}$	0.065 ± 0.007
$\boldsymbol{a_{xy}}$	$\boldsymbol{\frac{\partial M_{11}}{\partial y} + \frac{\partial M_{12}}{\partial x} - \frac{\partial \omega_z}{\partial x}}$	$\mathbf{2.306 \pm 0.002}$
a_{xz}	$\frac{\partial M_{11}}{\partial z} + \frac{\partial M_{13}}{\partial x} + \frac{\partial \omega_y}{\partial x}$	0.092 ± 0.005
a_{yz}	$\frac{\partial M_{13}}{\partial y} + \frac{\partial \omega_y}{\partial y} + \frac{\partial M_{12}}{\partial z} - \frac{\partial \omega_z}{\partial z}$	-0.122 ± 0.006
$\boldsymbol{b_{xx}}$	$\boldsymbol{\frac{\partial M_{12}}{\partial x} + \frac{\partial \omega_z}{\partial x}}$	$\mathbf{-1.811 \pm 0.001}$
$\boldsymbol{b_{yy}}$	$\boldsymbol{\frac{\partial M_{22}}{\partial y}}$	$\mathbf{-1.115 \pm 0.001}$
$\boldsymbol{b_{zz}}$	$\boldsymbol{\frac{\partial M_{23}}{\partial z} - \frac{\partial \omega_x}{\partial z}}$	$\mathbf{-8.727 \pm 0.006}$
b_{xy}	$\frac{\partial M_{22}}{\partial x} + \frac{\partial M_{12}}{\partial y} + \frac{\partial \omega_z}{\partial y}$	0.117 ± 0.002
b_{xz}	$\frac{\partial M_{23}}{\partial x} - \frac{\partial \omega_x}{\partial x} + \frac{\partial M_{12}}{\partial z} + \frac{\partial \omega_z}{\partial z}$	0.273 ± 0.004
b_{yz}	$\frac{\partial M_{23}}{\partial y} - \frac{\partial \omega_x}{\partial y} + \frac{\partial M_{22}}{\partial z}$	0.332 ± 0.005
c_{xx}	$\frac{\partial M_{13}}{\partial x} - \frac{\partial \omega_y}{\partial x}$	0.037 ± 0.001
c_{yy}	$\frac{\partial M_{23}}{\partial y} + \frac{\partial \omega_x}{\partial y}$	-0.036 ± 0.001
c_{zz}	$\frac{\partial M_{33}}{\partial z}$	-0.181 ± 0.004
c_{xy}	$\frac{\partial M_{23}}{\partial x} + \frac{\partial \omega_x}{\partial x} + \frac{\partial M_{13}}{\partial y} - \frac{\partial \omega_y}{\partial y}$	0.027 ± 0.001
c_{xz}	$\frac{\partial M_{33}}{\partial x} + \frac{\partial M_{13}}{\partial z} - \frac{\partial \omega_y}{\partial z}$	0.060 ± 0.003
c_{yz}	$\frac{\partial M_{33}}{\partial y} + \frac{\partial M_{23}}{\partial z} + \frac{\partial \omega_x}{\partial z}$	0.312 ± 0.004
		km s^{-1} kpc^{-2}

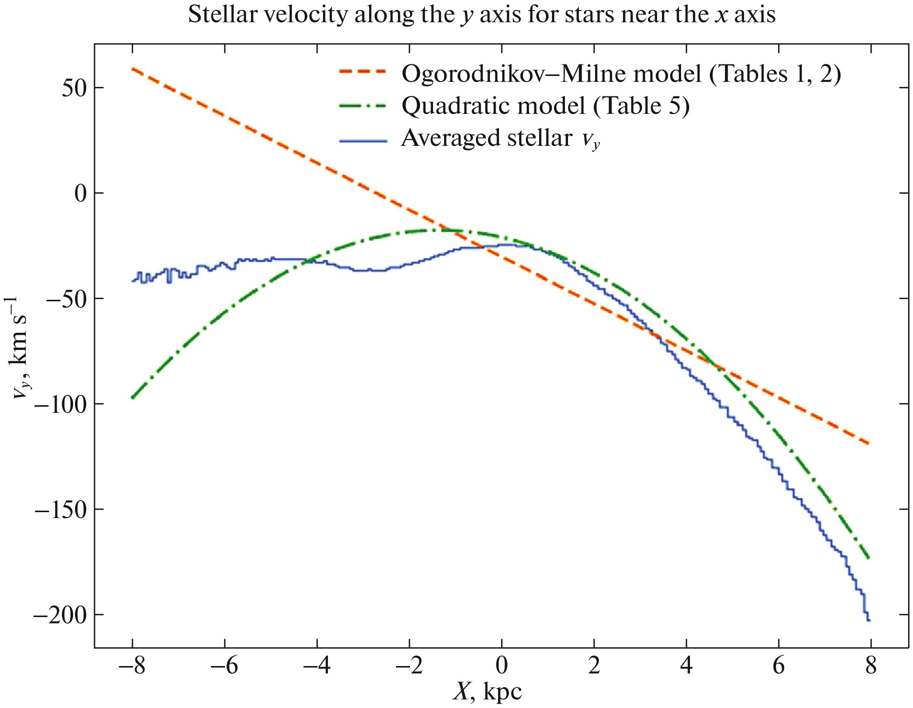

Fig. 9. Change of the component in the stellar velocities along the X axis (directed to the Galactic center). The averaging was performed for stars with $|y| < 10$ kpc and $|z| < 10$ kpc. The theoretical values of the velocity derived from the Ogorodnikov–Milne model and the quadratic model are shown on the graph.

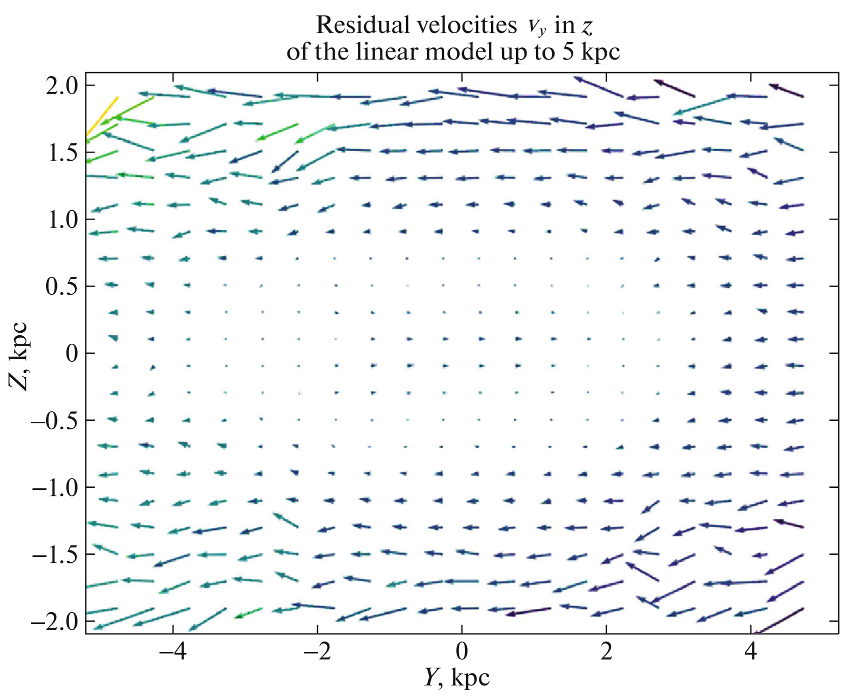

Fig. 10. Residual velocities of the linear model dv_x, dv_y, dv_z for stars with $|x| < 100$ pc in projection onto YZ (normal to the Galactic center direction).

If we depart from the formal mathematic point of view and turn to the physical picture, then, obviously, there is no need to determine all 27 derivatives. Note that, apart from the solar motion parameters U, V, and W, only two quantities remain significant in the Ogorodnikov–Milne model: ω_z and M_{12} (which are known in the simpler Oort–Lindblad model as the Oort parameters B and A). It is reasonable (at least the first time) to restrict our analysis to the derivatives of only these quantities.

In this case, the following values can be unambiguously determined from the quadratic coefficients:

$$\frac{\partial M_{12}}{\partial x} = 0.247 \pm 0.002 \text{ km s}^{-1} \text{ kpc}^{-2},$$
$$\frac{\partial \omega_z}{\partial x} = -2.059 \pm 0.002 \text{ km s}^{-1} \text{ kpc}^{-2}.$$

The partial derivatives of the Oort parameters along the direction to the Galactic center turn out to be significant. If this change in the linear model is ignored, then these effects penetrate into the solar motion parameter V, leading to its noticeable increase when it is determined from distant stars. Including the quadratic terms brings the parameter V with a value of 30.8 km s^{-1} to its usual value of 21.6 km s^{-1}. The adequacy of this extension of the standard model can be illustrated by the dependence of the stellar velocity (the component v_y) at various distances from the Galactic center. It is clearly seen from Fig. 9 that the traditional linear Ogorodnikov–Milne model is able to describe the motion of stars in the region from 0 to 3–4 kpc, whereas the actual stellar velocities (blue line) are far from the linear law; the quadratic approximation describes them much better. Similar conclusions are also reached by other researchers (see, e.g., Bobylev and Bajkova 2022).

Attention should also be paid to the significant coefficient $\partial M_{23}/\partial z - \partial \omega_x/\partial z$ at z^2 in the equation for v_y. This parameter is responsible for the nonlinear vertical change in velocity v_y (Fig. 10). Vityazev et al. (2012) have already analyzed the asymmetry in the kinematics of stars in the northern and southern Galactic hemispheres and considered the model of "stratified" Galactic rotation.

The coefficient $\partial M_{22}/\partial y$ at y^2 responsible for the change of the component as the y coordinate changes also shows significance.

CONCLUSIONS

We formulated the linear and quadratic stellar-kinematics models in the rectangular Galactic coordinate system. We found a relationship of all coefficients to the standard stellar-kinematics parameters and their partial derivatives in all three directions. The quadratic model was shown to have a considerably larger spatial region of applicability. We explained the behavior of the kinematic parameters of the linear Ogorodnikov–Milne model as a function of the distances to the stars being used. Four significant second-order effects were found:

(1) $\dfrac{\partial M_{12}}{\partial x} = 0.247 \pm 0.002$ km s^{-1} kpc^{-2} is the change in the velocity field deformation parameter in the XY plane (the Oort parameter A) with distance in the Galactic center–anticenter direction;

(2) $\dfrac{\partial \omega_z}{\partial x} = -2.059 \pm 0.002$ km s^{-1} kpc^{-2} is the vertical gradient of the angular velocity of rotation of the set of stars (the Oort parameter B);

(3) $\dfrac{\partial M_{22}}{\partial y} = -1.115 \pm 0.001$ km s^{-1} kpc^{-2} is the change in the parameter responsible for the contraction/extension of the system of stars in the XZ plane with distance along the direction of Galactic rotation;

(4) $\dfrac{\partial M_{23}}{\partial z} - \dfrac{\partial \omega_x}{\partial z} = -8.727 \pm 0.006$ km s^{-1} kpc^{-2} is a linear combination of inseparable parameters, the difference of the vertical gradient of the rotation velocity along the axis directed to the Galactic center and the velocity field deformation in the YZ plane.

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