Killing tensors associated with symmetric spaces

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Research goal

Question on the existence of quadratic integrals of motion for Hamiltonians of natural type

$$H=\frac{1}{2}\sum_{i=1} p_i^2+V(q_1,\dots,q_n)\,,$$

has been discussed for quite a long time. Most of the classical and modern works first study the existence of integrable geodesic flows at $V=0$ or the question of equivalent metrics. After that, one describes a class of potentials that can be added to a given geodesic flow while preserving the integrability property. If we abandon this usual sequence of actions, one can construct quadratic conservation laws for a sufficiently broad class of natural-type Hamiltonians describing motion in Euclidean space. In this paper we consider the Killing tensors and integrals of motion for systems related to the hierarchy of multicomponent nonlinear Schrödinger equations.

Objectives and research methodology

Let A and B be nondegenerate symmetric tensor fields of second order on the Euclidean space $\mathbb{R}^n$. If the Schouten bracket between them is zero $[\![A,B]\!]=0$, and the spectral problem $(A-\lambda B)\psi=0$ has n different real eigenvalues and the eigenvectors corresponding to them are normal, then A and B give rise to a n-dimensional linear space of tensor fields of second order, which are in involution and have common eigenvectors. This allows us to compute n independent second-order polynomials on momentum on the cotangent stratification T^*R^n

$$T_1=\sum_{ij} A^{ij}p_ip_j\,,\dots,T_n=\sum_{ij} K_n^{ij}p_ip_j.$$

In local coordinates the condition of integrability in terms of elements of the Haantjes tensor has the form of equality to zero

$$H^i_{jk}=K^i_\alpha K^\alpha_\beta N^\beta_{jk}+N^i_{\alpha\beta}K^\alpha_jK^\beta_k-K^i_\alpha\left(N^\alpha_{\beta k}K^\beta_j+N^\alpha_{j\beta}K^\beta_k\right).$$

Thus, we can show that in our case for the systems under consideration the spectral problem does not have the necessary number of real simple eigenvalues and normal eigenvectors, with the help of zero-torsion Haantjes.

In our case $A=g$ – standard metric in $\mathbb{R}^n$ and $B=K$ – Killing tensor satisfying the Killing equation

$$\nabla_iK^{jk}+\nabla_jK^{ki}+\nabla_kK^{ij}=0\,,$$

where ∇ – Levy-Chevita connectivity for g.

In Euclidean space the second-order Killing tensor K in the general case has the form

$$\sum_{i,j} a_{ij}X_i\circ X_j+\sum_{i,j,k} b_{ijk}X_i\circ X_{jk}+\sum_{i,j,k,m} c_{ijkm}X_{ij}\circ X_{km}\,,$$

where $X_i=\partial_i, X_{ij}=q_iX_j-q_jX_i$ – the basis of shifts and rotations in Euclidean space, a_{ij},b_{ijk},c_{ijkm} – arbitrary parameters and $\circ$ denotes the symmetric product.

Thus, we can find all valence tensors two associated with the Hamiltonian $H=T+V$ by solving the equation $d(KdV)=0$,

$$(KdV)_\alpha=g_{\alpha,\beta}K^{\beta,\gamma}\partial_\gamma V.$$

Results

In the case of systems associated with a hierarchy of multicomponent nonlinear Schrödinger equations, the Lax matrix is of the form

$$L(\lambda)=\lambda^2\mathcal{A}+\lambda\sum_\alpha q^\alpha(e_\alpha-e_{-\alpha})-$$
$$-\frac{1}{a}\sum_\alpha \mathrm{g}^{\alpha,-\alpha}p_\alpha(e_\alpha+e_{-\alpha})+\frac{1}{a}\sum_{\alpha,\beta} q_\alpha q_\beta[e_\alpha,e_{-\beta}]$$
$$+\Lambda.$$

The corresponding Hamiltonian

$$H=\frac{1}{4}\mathrm{Tr}(L^2(\lambda))\Big|_{\lambda=0}=\frac{1}{2}\sum_\alpha \mathrm{g}^{\alpha,-\alpha}p_\alpha^2-$$
$$-\frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta}\mathcal{R}_{-\alpha,\beta,\gamma,-\delta}q^\alpha q^\beta q^\gamma q^\delta+\frac{1}{2}\sum_\alpha \omega_\alpha(q^\alpha)^2$$

- $\mathcal{A}$ – element of the Lie algebra $\mathfrak{g}$, defining the Cartan involution and the decomposition of the root system $\Delta=\Delta_0\cup\Delta_+\cup\Delta_-$, where
$$\Delta_0=\{\alpha\in\Delta,\ \alpha(\mathcal{A})=0\},$$
$$\Delta_\pm=\{\alpha\in\Delta,\ \alpha(\mathcal{A})=\pm a\}.$$
- $e_\alpha,e_{-\alpha}$ – the corresponding Weyl generators.
- Λ – constant matrix defining "frequencies" ω_α.
- Metric and curvature tensor
$$\mathrm{g}^{\alpha,\beta}=\langle e_\alpha,e_\beta\rangle,$$
$$\mathcal{R}_{\alpha,\beta,\gamma,\delta}=\langle[e_\alpha,e_\beta],[e_\gamma,e_\delta]\rangle.$$

Proposition

Associated with Hermite symmetric spaces of type A.III, Newton's equations of motion

$$\ddot q^\alpha=\sum_{\beta,\gamma,\delta}\mathcal{R}^\alpha_{\beta,\gamma,-\delta}q^\beta q^\gamma q_\delta-\omega^\alpha q^\alpha,$$
$$\alpha,\beta,\gamma,\delta=1,\dots,N$$

in $\mathbb{R}^{mn}$ space have only $n+m-1$ independent quadratic integrals of motion in involution.

Example: $\mathbb{R}^4, A.III=SU(4)/S(U(2)\times U(2))$

The Hamiltonian:

$$H=\frac{p_1^2}{2}+\frac{p_2^2}{2}+\frac{p_3^2}{2}+\frac{p_4^2}{2}+\frac{1}{2}(q_1^2+q_2^2)^2+$$
$$+\frac{1}{2}(q_3^2+q_4^2)^2+(q_1q_3+q_2q_4)^2+$$
$$+\frac{a_1-b_1}{2}q_1^2+\frac{a_1-b_2}{2}q_2^2+\frac{a_2-b_1}{2}q_3^2+\frac{a_2-b_2}{2}q_4^2.$$

Two sets of second order integrals of motion:

$$f_1=-\frac{M_{12}^2}{a_1-a_2}+p_1^2+p_2^2+v_1,$$
$$f_2=\frac{M_{12}^2}{a_1-a_2}+p_3^2+p_4^2+v_2;$$

and

$$F_1=\frac{N_{12}^2}{b_1-b_2}+p_1^2+p_3^2+V_1,$$
$$F_2=-\frac{N_{12}^2}{b_1-b_2}+p_2^2+p_4^2+V_2.$$

In the example M_{12} and N_{12} describe two independent rotations in $\mathbb{R}^4$ and have the form

$$M_{12}=J_{1,3}+J_{2,4}=q_1p_3-q_3p_1+q_2p_4-q_4p_2,$$
$$N_{12}=J_{1,2}+J_{3,4}=q_1p_2-q_2p_1+q_3p_4-q_4p_3,$$
$$\{M_{12},N_{12}\}=0.$$

We can obtain another functionally independent quadratic integral by using a combination of fourth degree integrals and taking the integral of higher degree for integrability.

Conclusions

In this paper, quadratic conservation laws for Newton's equations have been constructed using the well-known Lax representation. The corresponding Killing tensors are associated with special linear combinations of basis rotations with respect to coordinate axes, forming a representation of the rotation subalgebra, and sequences of shifts along these axes. For example, in four-dimensional Euclidean space, right and left isocline rotations (Clifford shifts), which are classical objects in Euclidean geometry and the theory of Clifford algebras, are used to construct integrals of motion. At the same time it has been shown that in spite of non-zero Haantjes torsion these systems have a complete set of integrals of motion.

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