An Algorithm for Solving Two-Sided Linear Vector Equations in Tropical Algebra

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Abstract. We consider a two-sided vector equation that is defined in terms of tropical algebra as $\boldsymbol{Ax} = \boldsymbol{By}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are given matrices, $\boldsymbol{x}$ and $\boldsymbol{y}$ are unknown vectors. We propose a new procedure to solve this equation, which is based on the minimization of the distance between vectors of tropical vector spaces generated by the columns of the given matrices. The procedure produces a pair of vectors that provide the minimum distance between the spaces. If the two-sided equation has nontrivial solutions, the obtained vectors present a solution. Otherwise, these vectors compose a pseudo-solution that minimizes the deviation between both sides of the equation.

Introduction

We consider a vector equation that is defined in terms of tropical algebra as

$$\boldsymbol{Ax} = \boldsymbol{By},$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are given matrices, $\boldsymbol{x}$ and $\boldsymbol{y}$ are unknown vectors. This equation has unknowns on both sides and is usually referred to as the two-sided equation.

In tropical (idempotent) algebra, which deals with the theory and application of semirings and semifields with idempotent addition [1, 2, 3, 4, 5], solving the two-sided equation is a challenging problem from both analytical and the numerical perspectives. Since the first works of P. Butkovič [6, 7, 8] on the two-sided linear vector equation, the solution of this equation is still a topic of interest as can be seen in more recent papers [9, 10, 11, 12]. Existing approaches usually employ computational procedures based on iterative algorithms (see an overview of the state-of-art on the solution techniques given in Ch. 7 of [4]).

Available solution methods and techniques include combinatorial reduction algorithms [7], the elimination method [8], an algorithm of residuated functions for partially ordered sets [13], the alternating method [9], a combinatorial algorithm for the equation in the rational case [14], the method of bivariate equations and inequalities [11], methods for the equation with square matrices of special type [12] and others. However, these approaches often turn out to be not efficient enough for practical problems because of high computational complexity, rather restrictive assumptions or for other reasons. Therefore, the development of new methods that are able to supplement and complement existing approaches to solving the two-sided equation under consideration seems to be a rather urgent work.

In this paper, we propose a new procedure to solve the two-sided equation by minimization of the distance between vectors of tropical vector spaces generated by the columns of the given matrices. The procedure produces a pair of vectors that minimize the distance between the spaces. If the equation has nontrivial solutions, the obtained vectors present a solution. Otherwise, these vectors compose a pseudo-solution that minimizes the deviation between both sides of the equation.

The execution of the procedure consists in constructing a sequence of vectors that are pseudo-solutions of the two-sided equation in which the left and right sides are alternately replaced by constant vectors. Unlike the alternating algorithm [9], in which the corresponding inequalities are solved one by one instead of equations, the proposed procedure uses a different argument, looks simpler, and allows one to establish natural criteria for completing calculations. If the equation has no solutions, the procedure also finds a pseudo-solution and determines the value of the error associated with it, which can be useful in solving approximation problems.

1. Preliminary Definitions and Notation

In this section, we present basic definitions and notation to provide a formal framework for the description and solution of the two-sided linear equation in the tropical algebra setting. Further details on the theory and applications of tropical algebra can be found in a range of works, including [1, 2, 3, 4, 5].

1.1. Idempotent Semifield

Consider a set $\mathbb{X}$ that is closed under addition $\oplus$ and multiplication $\otimes$, and it includes the zero $\mathbb{0}$ and the identity $\mathbb{1}$. Assume that $(\mathbb{X}, \oplus, \mathbb{0})$ is an idempotent commutative monoid, $(\mathbb{X} \setminus \{\mathbb{0}\}, \otimes, \mathbb{1})$ is an Abelian group, and multiplication $\otimes$ distributes over addition $\oplus$. The algebraic structure $(\mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is usually referred to as an idempotent semifield.

In the semifield, addition is idempotent and multiplication is invertible: for each $x \in \mathbb{X}$ the equality $x \oplus x = x$ holds, and if $x \neq \mathbb{0}$, there exists an inverse x^{-1} such that $xx^{-1} = \mathbb{1}$ (here and hereafter the multiplication sign $\otimes$ is suppressed).

The power notation with integer exponents is thought of in the sense of the multiplication $\otimes$. Additionally, it is assumed that equation $x^p = a$ has a unique solution x for any $a \in \mathbb{X}$ and integer $p > 0$ to allow for the powers with rational exponents, which makes the semifield algebraically closed.

Addition induces a partial order: $x \leq y$ if and only if $x \oplus y = y$. It is assumed that this order extends to a total order, which makes the semifield linearly ordered.

An example of linearly ordered algebraically closed idempotent semifields is the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +)$, also known as $(\max,+)$-algebra. It has addition defined as max, multiplication as $+$, zero as $-\infty$ and identity as 0. The power x^y corresponds to the arithmetic product xy. The inverse x^{-1} of any $x \neq \mathbb{0}$ coincides with the opposite number $-x$. The order relation induced by addition agrees with the natural linear order on $\mathbb{R}$.

As another example, one can consider $\mathbb{R}_{\min} = (\mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times)$ (min-algebra), where $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$. It is equipped with the operations $\oplus = \min$ and $\otimes = \times$, which have the neutral elements $\mathbb{0} = +\infty$ and $\mathbb{1} = 1$. The notions of powers and inverses have the standard meaning. The partial order associated with addition is opposite to the natural linear order on $\mathbb{R}$.

1.2. Algebra of Matrices and Vectors

Let $\mathbb{X}^{m\times n}$ be the set of matrices over $\mathbb{X}$ with m rows and n columns. A matrix with all entries equal to $\mathbb{0}$ is the zero matrix. A matrix without zero rows and columns is called regular. A square matrix with the entries equal to $\mathbb{1}$ on the diagonal and to $\mathbb{0}$ elsewhere is the identity matrix.

Addition and multiplication of matrices and multiplication of matrices by scalars follow the standard entrywise rules with the arithmetic addition and multiplication replaced by $\oplus$ and $\otimes$.

A matrix that consists of one column (row) is a column (row) vector. All vectors are considered column vectors unless otherwise indicated. The set of column vectors with n elements is denoted by $\mathbb{X}^n$. A vector with all elements equal to $\mathbb{0}$ is the zero vector. A vector is called regular if it has no zero element.

For any nonzero column vector $\boldsymbol{x} = (x_i)$, a multiplicative conjugate row-vector $\boldsymbol{x}^- = (x_i^-)$ is defined, where $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$, and $x_i^- = \mathbb{0}$ otherwise.

1.3. Tropical Vector Space

Consider a system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{X}^m$. A vector $\boldsymbol{b} \in \mathbb{X}^m$ is a linear combination of these vectors if $\boldsymbol{b} = x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n$ for some $x_1, \ldots, x_n \in \mathbb{X}$. The set of linear combinations $\mathcal{A} = \{x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n |\ x_1, \ldots, x_n \in \mathbb{X}\}$ is closed under vector addition and scalar multiplication and referred to as the tropical vector space generated by the system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

For any vector $\boldsymbol{a} = (a_i)$ in a tropical space $\mathcal{A}$, consider its support given by $\operatorname{supp}(\boldsymbol{a}) = \{i |\ a_i \neq \mathbb{0},\ 1 \leq i \leq m\}$. For any nonzero vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ such that $\operatorname{supp}(\boldsymbol{a}) = \operatorname{supp}(\boldsymbol{b})$, we define the distance function

$$d(\boldsymbol{a}, \boldsymbol{b}) = \bigoplus_{i\in\operatorname{supp}(\boldsymbol{a})} \left(b_i^{-1}a_i \oplus a_i^{-1}b_i\right) = \boldsymbol{b}^-\boldsymbol{a} \oplus \boldsymbol{a}^-\boldsymbol{b}.$$

If $\operatorname{supp}(\boldsymbol{a}) \neq \operatorname{supp}(\boldsymbol{b})$, we consider the function to take a value greater that any $x \in \mathbb{X}$ and write $d(\boldsymbol{a}, \boldsymbol{b}) = \infty$. If $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$, then we set $d(\boldsymbol{a}, \boldsymbol{b}) = \mathbb{1}$.

In the context of the semifield $\mathbb{R}_{\max,+}$ where $\mathbb{1} = 0$, the function d coincides for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ with the Chebyshev metric

$$d_\infty(\boldsymbol{a}, \boldsymbol{b}) = \max_{1\leq i\leq m} \max(a_i - b_i, b_i - a_i) = \max_{1\leq i\leq m} |b_i - a_i|.$$

For an arbitrary idempotent semifield $\mathbb{X}$, the function d can be considered as a generalized metric with values in the set $[\mathbb{1}, \infty)$.

2. Distance Between Vectors and Solution of Equations

Let $\mathcal{A}$ be a tropical vector space generated by nonzero vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{X}^m$. Any vector $\boldsymbol{a} \in \mathcal{A}$ can be represented as a linear combination $\boldsymbol{a} = x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n$ with coefficients $x_1, \ldots, x_n \in \mathbb{X}$ and hence as the matrix-vector product $\boldsymbol{a} = \boldsymbol{A}\boldsymbol{x}$ with the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and the vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$.

The distance from a vector $\boldsymbol{b}$ to the vector space $\mathcal{A}$ is given by

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{a} \in \mathcal{A}} d(\boldsymbol{a}, \boldsymbol{b}) = \min_{\boldsymbol{x} \in \mathbb{X}^n} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \min_{\boldsymbol{x} \in \mathbb{X}^n} (\boldsymbol{b}^- \boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^- \boldsymbol{b}).$$

If the vector $\boldsymbol{b}$ is regular, then the minimum over all $\boldsymbol{x} \in \mathbb{X}$ on the right-hand side can be replaced by the minimum over only regular $\boldsymbol{x}$ (see, e.g. [15]) to write

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{x} > \boldsymbol{0}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}).$$

As it is easy to see, the equality $d(\mathcal{A}, \boldsymbol{b}) = \mathbb{1}$ corresponds to the condition $\boldsymbol{b} \in \mathcal{A}$, while the inequality $d(\mathcal{A}, \boldsymbol{b}) \neq \mathbb{1}$ means that $\boldsymbol{b} \notin \mathcal{A}$.

Suppose $\boldsymbol{A} \in \mathbb{X}^{m \times n}$ is a regular matrix, and $\boldsymbol{b} \in \mathbb{X}^m$ is a regular vector. Define the function

$$\Delta_{\boldsymbol{A}}(\boldsymbol{b}) = (\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^-)^- \boldsymbol{b}.$$

The next result is obtained in [16] (see also [15]).

Lemma 1. *Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ a regular vector. Then*

$$\min_{\boldsymbol{x} > \boldsymbol{0}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})},$$

where the minimum is achieved at $\boldsymbol{x} = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}(\boldsymbol{b}^- \boldsymbol{A})^-$.

If $\mathcal{A}$ is a vector space generated by the columns of the matrix $\boldsymbol{A}$, then the distance from a vector $\boldsymbol{b}$ to $\mathcal{A}$ is calculated as $d(\mathcal{A}, \boldsymbol{b}) = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}$. The vector in $\mathcal{A}$, which is closest to $\boldsymbol{b}$, takes the form $\boldsymbol{y} = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^-$.

Note that the condition $\boldsymbol{b} \in \mathcal{A}$ leads to the equality $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) = \mathbb{1}$, while $\boldsymbol{b} \notin \mathcal{A}$ to the inequality $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) > \mathbb{1}$.

Consider the problem to find regular vectors $\boldsymbol{x}$ that solve the equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{1}$$

The equation has the unknown vector on one side and hence is called one-sided.

The solution of equation (1) can be described as follows [16, 15].

Theorem 2. *Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ a regular vector. Then:*

1. *If $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) = \mathbb{1}$, then equation* (1) *has regular solutions; the vector $\boldsymbol{x} = (\boldsymbol{b}^- \boldsymbol{A})^-$ is the maximal solution.*
2. *If $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) > \mathbb{1}$, then no regular solution exists; the vector $\boldsymbol{x} = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}(\boldsymbol{b}^- \boldsymbol{A})^-$ is the best approximate solution in the sense of the distance function d.*

Note that $\sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}$ has the meaning of the minimum achievable deviation between the left and right sides of (1), measured on the scale of the function d.

Suppose $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{X}^{m\times k}$ are given regular matrices, and $\boldsymbol{x} \in \mathbb{X}^{n}$ and $\boldsymbol{y} \in \mathbb{X}^{k}$ are unknown regular vectors. Let us examine the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{y}. \tag{2}$$

Let $\mathcal{A}$ be the tropical vector space generated by the columns of $\boldsymbol{A}$, and $\mathcal{B}$ the space generated by the columns of $\boldsymbol{B}$. Define the distance between the spaces

$$d(\mathcal{A},\mathcal{B}) = \min_{\boldsymbol{a}\in\mathcal{A},\boldsymbol{b}\in\mathcal{B}} d(\boldsymbol{a},\boldsymbol{b}) = \min_{\boldsymbol{x}>\boldsymbol{0},\boldsymbol{y}>\boldsymbol{0}} d(\boldsymbol{A}\boldsymbol{x},\boldsymbol{B}\boldsymbol{y}).$$

The equality $d(\mathcal{A},\mathcal{B}) = \mathbb{1}$ means that the spaces $\mathcal{A}$ and $\mathcal{B}$ have nonempty intersection, and hence equation (2) has a solution $(\boldsymbol{x},\boldsymbol{y})$. If the distance satisfies the inequality $d(\mathcal{A},\mathcal{B}) > \mathbb{1}$, then the spaces have no common point (and thus the equation has no solution), while its value shows the minimum distance between vectors of the spaces (minimal deviation between both sides of the equation).

3. Solution Procedure for Two-Sided Equation

Consider a solution procedure that constructs a sequence of vectors from the spaces $\mathcal{A}$ and $\mathcal{B}$. The vectors are taken alternatively in both spaces so that after selecting a vector in one space, the next vector is found in the other space to minimize the distance between this space and the former vector. The vectors found in each space $\mathcal{A}$ and $\mathcal{B}$ are determined by coefficients in their decompositions as linear combinations of columns in the respective matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

Let $\boldsymbol{x}_0 \in \mathbb{X}^{n}$ be a regular vector and $\boldsymbol{a}_0 = \boldsymbol{A}\boldsymbol{x}_0 \in \mathcal{A}$. By applying Theorem 2, we find the minimum distance from the vector $\boldsymbol{a}_0$ to the vectors in $\mathcal{B}$ to be

$$d(\boldsymbol{a}_0,\mathcal{B}) = \sqrt{\Delta_0}, \qquad \Delta_0 = \Delta_{\boldsymbol{B}}(\boldsymbol{A}\boldsymbol{x}_0) = (\boldsymbol{B}((\boldsymbol{A}\boldsymbol{x}_0)^{-}\boldsymbol{B})^{-})^{-}\boldsymbol{A}\boldsymbol{x}_0.$$

This minimum distance is attained at a vector $\boldsymbol{b}_1 \in \mathcal{B}$ that is given by

$$\boldsymbol{b}_1 = \boldsymbol{B}\boldsymbol{y}_1, \qquad \boldsymbol{y}_1 = \sqrt{\Delta_0}((\boldsymbol{A}\boldsymbol{x}_0)^{-}\boldsymbol{B})^{-}.$$

The minimum distance from $\boldsymbol{b}_1$ to the vectors in $\mathcal{A}$ is equal to

$$d(\boldsymbol{b}_1,\mathcal{A}) = \sqrt{\Delta_1}, \qquad \Delta_1 = \Delta_{\boldsymbol{A}}(\boldsymbol{B}\boldsymbol{y}_1) = (\boldsymbol{A}((\boldsymbol{B}\boldsymbol{y}_1)^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{B}\boldsymbol{y}_1$$

and is achieved for a vector $\boldsymbol{a}_2 \in \mathcal{A}$ such that

$$\boldsymbol{a}_2 = \boldsymbol{A}\boldsymbol{x}_2, \qquad \boldsymbol{x}_2 = \sqrt{\Delta_1}((\boldsymbol{B}\boldsymbol{y}_1)^{-}\boldsymbol{A})^{-}.$$

In the same way, we obtain the distance $d(\boldsymbol{a}_2,\mathcal{B})$ by calculating Δ_2, which is then used for finding the vectors $\boldsymbol{y}_3$ and $\boldsymbol{b}_3$. Next, we calculate Δ_3 to evaluate the distance $d(\boldsymbol{b}_3,\mathcal{A})$, and find the vectors $\boldsymbol{x}_4$ and $\boldsymbol{a}_4$.

We continue the procedure to form a sequence of vectors $\boldsymbol{a}_0, \boldsymbol{b}_1, \boldsymbol{a}_2, \boldsymbol{b}_3, \boldsymbol{a}_4, \ldots$ taken alternatively from $\mathcal{A}$ and $\mathcal{B}$ to minimize the distance between successive vectors. At the same time, a sequence of pairs $(\boldsymbol{x}_0,\boldsymbol{y}_1), (\boldsymbol{x}_2,\boldsymbol{y}_3), \ldots$ is generated that provides successive approximations to the solution of equation (2).

Let us examine the sequence $\Delta_0, \Delta_1, \Delta_2, \dots$ and first note that it is bounded from below since $\Delta_i \geq \mathbb{1}$ for all $i = 0, 1, 2 \dots$ Furthermore, after some algebra, we can verify that $\Delta_{i+1} \leq \Delta_i$, which says that the sequence is nonincreasing. As a result, we conclude that this sequence converges to a limit $\Delta_* \geq \mathbb{1}$.

We observe that each element of the last sequence represents the squared distance between a vector of one of the spaces $\mathcal{A}$ and $\mathcal{B}$ and the nearest vector in the other space. Therefore, the equality $\Delta_i = \mathbb{1}$ for some i means that the spaces $\mathcal{A}$ and $\mathcal{B}$ have nonempty intersection, while equation (2) has regular solutions. Moreover, if i is even, then the intersection contains the vector $\boldsymbol{a}_i = \boldsymbol{A}\boldsymbol{x}_i$, and the pair of vectors $(\boldsymbol{x}_i, \boldsymbol{y}_{i+1})$ is a solution of the equation. In the case when i is odd, the intersection contains $\boldsymbol{b}_i = \boldsymbol{B}\boldsymbol{y}_i$, and the pair $(\boldsymbol{x}_{i+1}, \boldsymbol{y}_i)$ is a solution.

Reaching the equality $\Delta_i = \mathbb{1}$ indicates that sequence $\Delta_0, \Delta_1, \dots$ converges to $\Delta_* = \Delta_i$, which can be used in numerical computations as a stop criterion for iterations. If $\mathcal{A}$ and $\mathcal{B}$ do not intersect, then the inequality $\Delta_* > \mathbb{1}$ holds, whereas equation (2) does not have regular solutions. In this case, the procedure stops as soon as there is a repeating element in any of the sequences $\boldsymbol{x}_0, \boldsymbol{x}_2, \dots$ or $\boldsymbol{y}_1, \boldsymbol{y}_3, \dots$

The above described solution procedure can be summarized as follows.

Algorithm 1. *Solution of the two-sided equation* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x}$*:*

1. *Input regular matrices* $\boldsymbol{A}, \boldsymbol{B}$*; set* $i = 0$*; fix a regular vector* $\boldsymbol{x}_0$*.*
2. *Calculate*
$$\Delta_i = (\boldsymbol{B}((\boldsymbol{A}\boldsymbol{x}_i)^- \boldsymbol{B})^-)^- \boldsymbol{A}\boldsymbol{x}_i, \qquad \boldsymbol{y}_{i+1} = \sqrt{\Delta_i}((\boldsymbol{A}\boldsymbol{x}_i)^- \boldsymbol{B})^-.$$
3. *If* $\Delta_i = \mathbb{1}$ *or* $\boldsymbol{y}_{i+1} = \boldsymbol{y}_j$ *for some* $j < i$*, then set*
$$\Delta_* = \Delta_i, \qquad \boldsymbol{x}_* = \boldsymbol{x}_i, \qquad \boldsymbol{y}_* = \boldsymbol{y}_{i+1},$$
and stop; otherwise set $i = i + 1$*.*
4. *Calculate*
$$\Delta_i = (\boldsymbol{A}((\boldsymbol{B}\boldsymbol{y}_i)^- \boldsymbol{A})^-)^- \boldsymbol{B}\boldsymbol{y}_i, \qquad \boldsymbol{x}_{i+1} = \sqrt{\Delta_i}((\boldsymbol{B}\boldsymbol{y}_i)^- \boldsymbol{A})^-.$$
5. *If* $\Delta_i = \mathbb{1}$ *or* $\boldsymbol{x}_{i+1} = \boldsymbol{x}_j$ *for some* $j < i$*, then set*
$$\Delta_* = \Delta_i, \qquad \boldsymbol{x}_* = \boldsymbol{x}_{i+1}, \qquad \boldsymbol{y}_* = \boldsymbol{y}_i,$$
and stop; otherwise set $i = i + 1$*.*
6. *Go to step* 2*.*

If upon completion of the algorithm, we have $\Delta_* = \mathbb{1}$, then equation (2) has regular solutions including the obtained pair of vectors $(\boldsymbol{x}_*, \boldsymbol{y}_*)$. In the case when $\Delta_* > \mathbb{1}$, the equation has no regular solution, while Δ_* indicates the minimum deviation between both sides of the equation, which is attained at $(\boldsymbol{x}_*, \boldsymbol{y}_*)$.

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