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DARK MATTER AS A GRAVITATIONAL EFFECT IN THE EMBEDDING THEORY APPROACH

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We discuss a possibility to explain observations usually related to the existence of dark matter by passing from the general relativity (GR) theory to a modified theory of gravity, the embedding theory proposed by Regge and Teitelboim. In this approach, it is assumed that our space–time is a four-dimensional surface in a ten-dimensional flat ambient space. This clear geometric interpretation of a change of a variable in the GR action leading to a new theory distinguishes this approach from the known alternatives: mimetic gravity and other variants. After the passage to the modified theory of gravity, additional solutions that can be interpreted as GR solutions with additional fictitious matter appear besides the solutions corresponding to GR. In that theory, one can try to see dark matter, with no need to assume the existence of dark matter as a fundamental object; its role is played by the degrees of freedom of modified gravity. In the embedding theory, the number of degrees of freedom of fictitious matter is sufficiently large, and hence an explanation of all observations without complicating the theory any further can be attempted.

Keywords: dark matter, embedding theory, modified theory of gravity, isometric embeddings

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1. Introduction

The mystery of the nature of dark matter (DM) is one of the most intriguing problems in modern fundamental physics. The hypothesis of its existence allows simultaneously explaining many contradictions appearing in interpreting observations, from the scale of galaxies to that of the Universe (see, e.g., [1]). On a galactic scale, this is the explanation of deviations from the expected motion of stars ("galaxy rotation curves"); on a large scale, this is the explanation of the results of observation of gravitational lensing and baryon acoustic oscillations. Finally, on a cosmological scale, this is the solution of the problem of structure formation and (along with dark energy) the role played in solving the problem of the total mass of the Universe compared with the value corresponding to the critical density.

On the whole, all observation data existing at present (among which the observation of CMB anisotropy plays the most important role) are described rather well by the ΛCDM model [2],[1] which is currently the standard cosmology model. In its framework, DM can be regarded as nonrelativistic dust-like matter, which produces the same gravitational field as ordinary matter, and the nongravitaional interaction of DM with ordinary matter is either absent or undetectably weak (see [3] for the main known DM properties following from the existing observations).

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[1] Λ-cold dark matter model.

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Numerous cases where the introduction of DM is useful makes its existence very probable, although attempts to directly detect it gave no results [4], [5]. Perhaps the most popular ideas to describe the DM nature are currently the assumptions that DM consists of weakly interacting massive particles (WIMPs) [6] or is the so-called *fuzzy* DM [7]. The failure of direct detection attempts is explained by the weakness of the coupling between DM and ordinary matter. Models in which DM has a self-action [8] are also considered among others, including attempted solutions to the problem of DM density at galaxy centers, which is too high—the so-called "core–cusp" problem [9].

However, the fact that it has not been possible to detect DM in any interaction besides the gravitational one suggests that it is can be an effect of the description of gravity rather than real matter, i.e., in fact, DM does not exists (from the standpoint of fundamental theory). Apparently, this was first done within the modified Newtonian dynamics (MOND) paradigm [10]. In the framework of this approach, the above-mentioned contradiction can be eliminated rather successfully on galactic or similar scales; but on a cosmological scale, the MOND paradigm does not work equally well [11].

Modifications of the theory of gravity that introduce additional degrees of freedom compared to general relativity (GR) seem much more promising. If we discuss solutions of such modified theories from the standpoint of GR, then additional gravitational degrees of freedom describe the dynamics of some fictitious matter that can be identified with DM. We emphasize that in this case, DM has independent dynamics, i.e., it can move in a way that is different from that of ordinary matter. Modified theories of gravity that have additional degrees of freedom compared to GR include, for example, $f(R)$ gravity, scalar–tensor theories of gravity, and a number of others (see [12]). We note that after passing to a modified theory of gravity, DM no longer exists as an independent fundamental object, and only when we discuss this theory in terms of GR does DM arises as a source of an additional contribution to the Einstein equations.

Among the modified theories of gravity with additional degrees of freedom, in the context of the description of DM, the *mimetic theory of gravity* [13] appears to have been discussed most often in the last decade. The name of the theory reflects the fact that its gravitational degrees of freedom "mimic" the presence of some fictitious matter. In the simplest version of the theory, this matter is dust-like and moves potentially, i.e., in a vortex-free manner. Mimetic gravity is obtained from GR by replacing the independent variable

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\gamma\delta}(\partial_\gamma\lambda)(\partial_\delta\lambda) \tag{1}$$

in the GR action

$$S = S^{\mathrm{EH}} + S_{\mathrm{m}}, \qquad S^{\mathrm{EH}} = -\frac{1}{2\varkappa}\int d^4x\,\sqrt{-g}\,R, \tag{2}$$

where S^{EH} is the Einstein–Hilbert action (we use the signature $+---$) and S_{m} is the action of ordinary matter. In (1), $\tilde{g}_{\mu\nu}$ is the auxiliary metric (and $\tilde{g}^{\mu\nu}$ is the metric inverse to it), which is a new independent variable along with the scalar field λ. Because the common factor $\tilde{g}_{\mu\nu}$, obviously, does not affect $g_{\mu\nu}$, ten independent field variables exist in the new theory, as originally in GR; this is usually referred to as "isolating the conformal mode" of the metric.

The variation of the action with respect to the new variables $\tilde{g}_{\mu\nu}$ and λ gives equations of motion that can be written as

$$G^{\mu\nu} = \varkappa(T^{\mu\nu} + nu^\mu u^\nu), \tag{3}$$

$$D_\mu(nu^\mu) = 0, \tag{4}$$

where $G^{\mu\nu}$ is the Einstein tensor, $T^{\mu\nu}$ is the energy–momentum tensor (EMT) of ordinary matter, and D_μ

is the covariant derivative, and the notation

$$n \equiv g_{\mu\nu}\left(\frac{1}{\varkappa}G^{\mu\nu} - T^{\mu\nu}\right), \tag{5}$$

$$u_\mu \equiv \partial_\mu \lambda \tag{6}$$

is used. It can be verified that the relation

$$g^{\mu\nu} u_\mu u_\nu = 1 \tag{7}$$

is satisfied identically. If this identity and notation (5) are taken into account, then among the ten Einstein equations in (3), one (obtained by contraction with $g_{\mu\nu}$) is satisfied identically, which corresponds to the invariance of the action under the Weyl transformation of the auxiliary metric $\tilde{g}_{\mu\nu}$.

However, the obtained equations can be interpreted somewhat differently by assuming that n is some additional dynamical variable obeying Eq. (4) and that relation (5) for it is the tenth Einstein equation (precisely the one that follows by contraction with $g_{\mu\nu}$). The physical meaning of this variable is easy to understand if we recall that $nu^\mu u^\nu$, which is contained in the right-hand side of Einstein equations (3) as the EMT of some additional matter, is the EMT of dust-like matter with the density n and velocity u^μ (take that the velocity normalization condition (7) into account). Thus, it turns out that mimetic gravity is completely equivalent to GR with additional fictitious matter that moves potentially (this follows from (6)). Interestingly, Eq. (4), which plays the role of the equation of motion for fictitious matter, turns out to be a corollary of Einstein equation (3), as is well known for dust-like matter.

As we can see, as a result of the change of a variable in the GR action, despite the number of independent variables remaining unchanged, new dynamical variables appear in the theory when interpreting it from the GR standpoint, namely, the density of fictitious matter n and the scalar λ parameterizing the potential velocity. The reason for this is the presence of the differentiation operation in (1); as a result of such a *differential change of variables*, the theory dynamics can be enriched [14]. As regards attempts to explain the effects associated with DM in the framework of such an approach, it is important to emphasize that, the fictitious matter described as a gravitational effect has its own dynamics (dynamical degrees of freedom) and hence its own initial data. Depending on the initial data, we can have solutions that are completely equivalent to GR (if we have $n = 0$ in some region of space at the initial instant, which allows, for example, describing the galaxies where DM is virtually absent) as well as solutions for which the fictitious matter moves quite differently from ordinary matter; this behavior of DM is known from gravitational lensing observations of the Bullet cluster [15].

For theories arising as a result of changing a variable in the GR action, there is usually a possibility of reformulating them in the GR form with additional fictitious matter on the level of not only the equations of motion but also the action. For mimetic gravity, an equivalent reformulation can be obtained by considering GR with an additional contribution to the action of the form [16]

$$S^{\text{add}} = -\frac{1}{2}\int d^4x\, \sqrt{-g}\, (1 - g^{\mu\nu}(\partial_\mu\lambda)(\partial_\nu\lambda))n, \tag{8}$$

with total action

$$S = S^{\text{EH}} + S_{\text{m}} + S^{\text{add}}. \tag{9}$$

Here, the independent variables are the usual metric $g_{\mu\nu}$ and two scalar fields n and λ describing fictitious matter, which still play the role of matter density and velocity potential. Most often, mimetic gravity is studied precisely in this formulation. There are other possibilities of choosing the contribution to the action of a potentially moving ideal fluid without pressure; various options were discussed in [17].

To reproduce the DM properties that are needed to explain the existing observations in the mimetic gravity framework, the theory has to be made somewhat complex. For example, the addition of a scalar field potential λ to the action has been considered, leading to the appearance of nonzero pressure in fictitious matter [18]. The possibility of adding a contribution with higher derivatives of this field has also been studied, turning fictitious matter into a nonideal liquid [18]–[20]. There is also a possibility to transform the contribution to action (8) such that the fictitious matter, while remaining dust-like, moves nonpotentially [17]. In the last case, in addition to the fields n and λ, another two scalar fields participate in the description of fictitious matter; it is then possible to return to the original formulation of mimetic gravity with the independent auxiliary metric $\tilde{g}_{\mu\nu}$ and three scalar fields contained in the corresponding expression for the physical metric $g_{\mu\nu}$, which is analogous to (1) (see [21] and the references therein for the current status of the mimetic gravity approach).

The fact that mimetic gravity is insufficiently "complex" (the fictitious matter corresponding to it has too simple dynamics and the theory has to be made more complex to explain the effects associated with DM) can be viewed as a disadvantage of the approach. Another possible disadvantage is the artificial form of the change of a variable in (1), which underlies the approach: it is difficult to formulate any physical or geometrical arguments in favor of just this type of replacement. Both these disadvantages are absent in another modified theory of gravity, which had appeared earlier and which also arises as a result of changing the variable in the GR action. This change of a variable

$$g_{\mu\nu} = (\partial_\mu y^a)(\partial_\nu y^b)\eta_{ab} \tag{10}$$

has a clear geometrical meaning: the new independent variable $y^a(x^\alpha)$ is a function of the embedding of the four-dimensional surface into the ambient space with a flat metric η_{ab} (the indices $a, b, \dots$ label the components of the Lorentz coordinates of the ambient space), and (10) defines the induced metric on the embedded surface. Thus, this modification of gravity is based on a simple assumption: our space–time is a four-dimensional surface in some flat pseudo-Euclidean ambient space. We note that in the GR framework, our space–time is considered an abstract pseudo-Riemannian space. Such a modified theory of gravity was proposed in 1975 [22] and is called Regge–Teitelboim gravity or the embedding theory. The idea of the approach was obviously suggested by the geometric description of strings, which was reflected in the title of original paper [22]: "*General relativity à la string: a progress report.*" The difference between the embedding theory and the Nambu–Goto string theory lies in the dimension under consideration ($1+3$ instead of $1+1$) and in the choice of the action: for the embedding theory, GR action (2) is used, and for the string, the volume (area in a two-dimensional space) of the manifold is used instead of the Einstein–Hilbert action.

As shown below, the embedding theory is sufficiently "complex" to explain the effects associated with DM in the framework of such an approach without further complicating the theory. In Secs. 2 and 3, we describe the embedding theory in more detail, including the possibility of formulating it in GR form with a contribution of additional fictitious matter to the action. In Sec. 4, we discuss the weak gravity limit for the embedding theory and show that to satisfy the superposition principle for the gravitational field, the background embedding corresponding to the flat metric must correspond to the generic case and, in particular, not be reducible to a four-dimensional plane in the ambient space. In Sec. 5, we present the results of considering the nonrelativistic limit of the embedding theory, in which fictitious matter behaves like a nonrelativistic fluid with some self-action. Further studies of the properties of this liquid and their comparison with the observed DM properties will allow deciding whether the passage to the description of gravity in the form of the embedding theory can explain, any additional modifications, the observed effects that are currently explained by the hypothesis stating the DM existence.

2. Embedding theory: Regge–Teitelboim gravity

When describing gravity in the form of an embedding theory, the embedding function $y^a(x^\alpha)$ in terms of which $g_{\mu\nu}(x^\alpha)$ is expressed by as an induced metric, Eq. (10), is an independent variable instead of the metric $g_{\mu\nu}(x^\alpha)$. A parameter characterizing this approach, the ambient space dimension N, then appears. In addition, in principle, there is some arbitrariness in the choice of its signature. If we assume that it be possible to define an arbitrary space–time metric in terms of the embedding theory (at least locally, i.e., for some part of space–time), then a restriction on N arises in accordance with the Janet–Cartan–Friedman theorem [23]. This theorem states that an arbitrary Riemannian space of dimension d can be locally isometrically embedded into any Riemannian space of the dimension

$$N \geq \frac{d(d+1)}{2}, \tag{11}$$

and hence, in particular, into the flat space of such a dimension. The theorem was generalized to the pseudo-Riemannian case by Friedman [23]; then in addition to restriction (11) on the total dimension N, an additional natural constraint arises, that the number of both spatial and temporal directions in the ambient space be not less than that in the embedded pseudo-Riemannian space.

Because $d = 4$ for our space–time, restriction (11) gives $N \geq 10$, and the embedding theory with the ambient space dimension $N = 10$ is considered most often (it is amazing that this is the same dimension in which superstring theory becomes consistent, but the reason for this coincidence is entirely unclear). This value is the most natural because the numbers of the old variables $g_{\mu\nu}$ and of the new variables y^a in the change of variables (10) then coincide. , The ambient space signature must be such that it have at least one timelike direction. Just one timelike direction is typically used; in this case, causality can be established in the ambient space: there are no closed timelike lines, which is important for the embedding theory in the form of field theory in the ambient space–time [24]. Thus, it turns out to be most natural to choose the ten-dimensional Minkowski space $R^{1,9}$ as the ambient space for the embedding theory.

As the action of the embedding theory, GR action (2) is taken, with the induced metric (10). Varying with respect to the new independent variable y^a yields the Regge–Teitelboim equations [22], which can be written in two equivalent forms,

$$D_\mu((G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\nu y^a) = 0 \tag{12}$$

or

$$(G^{\mu\nu} - \varkappa T^{\mu\nu})b^a_{\mu\nu} = 0, \tag{13}$$

where $b^a_{\mu\nu} = D_\mu \partial_\nu y^a$ is the second fundamental form of a surface. We note although the Einstein equations contained the second derivatives of the metric and change of variables (10) contains differentiation, Eqs. (13) contain no derivatives of y^a of an order higher than two. To see this, it suffices to use the formula [25]

$$G^{\mu\nu} = \frac{1}{2} g_{\xi\zeta} E^{\mu\xi\alpha\beta} E^{\nu\zeta\gamma\delta} b^a_{\alpha\gamma} b^b_{\beta\delta} \eta_{ab}, \tag{14}$$

where $E^{\mu\xi\alpha\beta} = \varepsilon^{\mu\xi\alpha\beta}/\sqrt{|g|}$ is the covariant unit antisymmetric tensor.

As we can see, the Regge–Teitelboim equations contain more solutions than the Einstein equations: besides the Einstein solutions, there are others. The resulting extension of the theory dynamics is a consequence of the presence of differentiations in the change of variables (10), just as in mimetic gravity discussed in Sec. 1. As a result, in the embedding theory, as in mimetic gravity, there are more dynamical variables than in GR, and these variables can be assumed to describe some fictitious matter, which can be identified with DM. To isolate these variables, it is convenient to rewrite the Regge–Teitelboim equations in the form of the equivalent set of equations [26]:

$$G^{\mu\nu} = \varkappa(T^{\mu\nu} + \tau^{\mu\nu}), \tag{15}$$

$$D_\mu(\tau^{\mu\nu}\partial_\nu y^a) = 0 \quad \Longleftrightarrow \quad \tau^{\mu\nu} b^a_{\mu\nu} = 0. \tag{16}$$

The first of them is the Einstein equation with an additional contribution of the EMT $\tau^{\mu\nu}$ of fictitious matter. The second can be viewed as an equation restricting the possible behavior of $\tau^{\mu\nu}$, and therefore as an equation of motion of this fictitious matter. Thus, we can assume that fictitious matter is described by the variables y^a and $\tau^{\mu\nu}$. Not all of these twenty variables are dynamical, i.e., have their own arbitrary initial data. It is difficult to determine the number of dynamical variables corresponding to fictitious matter and somehow isolate them in the general case, but this can be done in the nonrelativistic limit, which is discussed in Sec. 4. It is also possible to calculate the number of the degrees of freedom of fictitious matter by studying the canonical (Hamiltonian) formulation of the theory (see below).

As was already noted in Sec. 1, theories arising as a result of changing the variable in the GR action can usually be reformulated in the GR form with additional fictitious matter not only at the level of the equations of motion but also at the level of the action if the total action in form (9) is written with some S^{add}. This can also be done for the embedding theory in different ways, using which can be convenient in analyzing the properties of fictitious matter in different cases. The simplest way is to write S^{add} in the form [17]

$$S^{\text{add}} = \frac{1}{2}\int d^4x\,\sqrt{-g}((\partial_\mu y^a)(\partial_\nu y_a) - g_{\mu\nu})\tau^{\mu\nu}. \tag{17}$$

With this choice of S^{add}, the fictitious matter EMT $\tau^{\mu\nu}$ (which we assume to be symmetric) is a Lagrange multiplier, the variation over which yields the relation of the metric $g_{\mu\nu}$ to the embedding function y^a in accordance with condition (10). An alternative way of writing S^{add}, with a set of conserved currents chosen an independent variable instead of $\tau^{\mu\nu}$, is described in Sec. 3.

After the appearance of [22], the ideas of the embedding theory were critically discussed in [27]; subsequently, they were repeatedly used to describe gravity, including its quantization (see, e.g., [24], [28]–[32]). At first, the embedding theory was mainly regarded as a reformulation of GR that was potentially more convenient for quantization because of the presence of a flat ambient space; therefore, the presence of non-Einstein solutions in it was considered a disadvantage. To eliminate it, the proposal in [22] was to impose additional constraints making the theory equivalent to GR; the canonical formalism for the resultant theory was then studied. These studies were continued in [25], [33], [34], and the Hamiltonian description of the complete embedding theory was investigated in [35]–[39]. Such a description allows, in particular, determining the number of degrees of freedom of the embedding theory: it turns out to be equal to six, i.e., in comparison with GR, there are another four degrees of freedom corresponding to fictitious matter [39] (we mean four pairs of conjugate canonical variables, which in the Lagrangian language correspond to four variables whose time evolution is controlled by a second-order differential equation). With the appearance of the DM problem, interest in the study of the complete embedding theory increased because non-Einstein solutions can be used to explain the DM effects [40]–[42]. A somewhat outdated but very detailed list of references related to the embedding theory and similar problems can be found in [43].

3. Alternative form of the action

We note that in its first form, Eq. (16), which can be understood as the equation of motion of fictitious matter, expresses the conservation of a set of currents labeled with the subscript a,

$$\partial_\mu(\sqrt{-g}\,j^\mu_a) = 0, \tag{18}$$

where

$$j^\mu_a = \tau^{\mu\nu}\partial_\nu y_a. \tag{19}$$

The description of fictitious matter in the language of such a set of currents is useful for a better understanding of its properties. It is therefore interesting to write the action of fictitious matter with j^μ_a considered as one of the variables describing it instead of $\tau^{\mu\nu}$. We note that doing so is not obstructed by the fact that j^μ_a contains more components (40) than $\tau^{\mu\nu}$ does (10 components).

The corresponding action has the form [44]

$$S^{\rm add} = \int d^4x\,\sqrt{-g}\big(j^\mu_a\partial_\mu y^a - \operatorname{tr}\sqrt{g_{\mu\nu}j^\nu_a j^{\alpha a}}\,\big), \tag{20}$$

where a square root of a matrix with the indices μ and α is understood, followed by taking the trace (the operation tr). Fictitious matter is described, in addition to j^μ_a, also by the embedding function y^a, which becomes a Lagrange multiplier in this approach. Varying with respect to it gives the condition

$$D_\mu j^\mu_a = 0. \tag{21}$$

The variation with respect to j^μ_a leads to another equation of motion,

$$\partial_\mu y^a = \hat\beta_{\mu\nu} j^{\nu a}, \tag{22}$$

where the symmetric tensor $\hat\beta_{\mu\nu}$ is inverse to $\beta^{\mu\nu}$, whose components β^α_μ are defined as the square root of the matrix $g_{\mu\nu}j^\nu_a j^{\alpha a}$ with the indices μ and α. The square root if defined in terms of its Taylor expansion about the unit matrix (see [44] for more details). It is easy to verify that (22) implies the induced-metric condition (10)

$$(\partial_\mu y^a)(\partial_\nu y_a) = \hat\beta_{\mu\alpha} j^{\alpha a}\hat\beta_{\nu\beta} j^\beta{}_a = \hat\beta_{\mu\alpha}\hat\beta_{\nu\beta}\beta^{\alpha\gamma}\beta_\gamma{}^\beta = g_{\mu\nu}. \tag{23}$$

Because action (20) is a contribution to the total action (9), variation (20) with respect to $g_{\mu\nu}$ gives the EMT of fictitious matter $\tau^{\mu\nu}$. It can be shown [44] that $\tau^{\mu\nu} = \beta^{\mu\nu}$ is obtained as a result of the variation,. It is then possible to express j^μ_a from equation of motion (22), with the result coinciding with (19). In addition, because equation of motion (21) coincides with Eq. (19), we conclude that GR with additional contribution (20) to the action in (9) completely reproduces the equation of motion of the embedding theory.

It is useful to see how action (20) is simplified for the set of currents j^μ_a where all currents are zero except the one corresponding to $a = 0$ ($j^\mu_a = j^\mu\delta^a_0$); the same result can be obtained if the dimension of the ambient space is reduced to $N = 1$. For a matrix of unit rank (which is the rank of the matrix in the radicand in (20)), the trace of its square root coincides with square root of the trace, whence we obtain a simplified action in the form

$$\tilde S^{\rm add} = \int d^4x\,\sqrt{-g}\big(j^\mu\partial_\mu y^0 - \sqrt{j^\mu g_{\mu\nu}j^\nu}\,\big). \tag{24}$$

This expression is one of the possibilities to write the action of a potentially moving ideal fluid without pressure [17], which, as was mentioned in Sec. 1, is the fictitious matter of mimetic gravity. Therefore, it can be said that mimetic gravity is a certain limit of the embedding theory, and the complete embedding theory is more complex. However, if we restrict the class of fields in the action, the set of variations of these fields also decreases; this means that some of the equations of motion are lost. The result of a more accurate analysis in the nonrelativistic limit of the embedding theory is described in Sec. 5.

4. The weak gravity limit

The gravitational field is considered weak if the metric has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \ll 1, \tag{25}$$

where $\eta_{\mu\nu}$ is the Minkowski space metric. When describing gravity in the framework of the embedding theory, a question arises: how to choose the background embedding function $\bar{y}^a$ corresponding to the background metric $\eta_{\mu\nu}$? It is obvious that the choice of $\bar{y}^a$ involves arbitrariness, as is evidenced by the well-known fact that in three-dimensional space, a part of a cylinder has the same flat metric as a part of the plane.

The simplest choice of the background embedding function corresponds to the four-dimensional plane,

$$\bar{y}^a(x^\mu) = \delta^a_\mu x^\mu. \tag{26}$$

However, if solutions are sought in the form of small deviations from such a background, i.e., as

$$y^a = \bar{y}^a + \delta y^a, \tag{27}$$

then equations of motion (13) of the embedding theory are nonlinear in the deviations δy^a (they are cubic, as can be seen from the expression for the Einstein tensor in form (14)). Thus, the embedding theory is nonlinearized on background (26), which was already mentioned in [27].

In addition to the inconvenience in technical terms, there is a more significant problem, the nonlinearity of the equations of motion in the weak gravity limit contradicts the principle of superposition for the gravitational field. Indeed, for the correction $h_{\mu\nu}$ to the flat metric corresponding (in the sense of Regge–Teitelboim equation (13)) to the sum of two contributions to the EMT of ordinary matter to be equal to the sum of the corrections $h_{\mu\nu}$ corresponding to each of the contributions to the EMT, the factor $b^a_{\mu\nu}$ in (13) must be nonzero in the zeroth order of the expansion in terms of δy^a. This must in fact be the case for each of the six nontrivial equations (13): it must be taken into account that, the second fundamental form $b^a_{\mu\nu}$ is by definition orthogonal with respect to the index a to the four vectors $\partial_\mu y^a$ that are tangent to the surface, and hence, among the ten equations in (13), four are satisfied identically at each point. As a result, we conclude that superposition principle for the gravitational field requires that the second fundamental form $\bar{b}^a_{\mu\nu}$ corresponding to the background embedding function $\bar{y}^a$ have rank 6 if it is considered as a 10×10 matrix with the indices a and $(\mu\nu)$ (we note that $b^a_{\mu\nu}$ is symmetric with respect to the permutation of μ, ν, and this pair of indices can be replaced with a multi-index taking ten values). For trivial embedding (26), it turns out that $\bar{b}^a_{\mu\nu} = 0$, and hence this condition for the rank is not satisfied.

Embeddings for which the $b^a_{\mu\nu}$ have the highest possible rank (for the considered dimensions of the surface and the ambient space, it is equal to 6) can be called *unfolded*, because in this case, the surface in the ambient space occupies a subspace of the maximum possible dimension: it cannot be "unfolded" any more. This corresponds to the generic case because reducing the rank of $b^a_{\mu\nu}$ amounts to some additional condition and corresponds to a set of measure zero. The embeddings of flat metrics with the property of "unfoldedness" were studied in [45]; for spherical symmetry, such an embedding was proposed in [42].

If the background embedding function $\bar{y}^a$ is an unfolded one, then the relation between the corrections $h_{\mu\nu}$ to the flat metric and δy^a to the background embedding function is linear. To see this, we decompose an arbitrary quantity δy^a into the longitudinal and transverse parts with respect to the background surface:

$$\delta y^a = \xi^\mu \partial_\mu \bar{y}^a + \delta y^a_\perp, \qquad \delta y^a_\perp \partial_\mu \bar{y}_a = 0. \tag{28}$$

From (10), in the first order in δy^a, we then find

$$h_{\mu\nu} = (\partial_\mu \bar{y}_a)(\partial_\nu \delta y^a) + (\partial_\mu \delta y^a)(\partial_\nu \bar{y}_a) = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\bar{b}^a_{\mu\nu} \delta y_{a\perp}, \tag{29}$$

where the definition and properties of the second fundamental form are used (see, e.g., [25] for more details). If $\bar{b}^a_{\mu\nu}$ has rank 6, then all the 6 components $\delta y^a_\perp$ can be found from this relation, which means a linear relation between all components $h_{\mu\nu}$ and δy_a. In the case of a lower rank, it is impossible to find some components $\delta y^a_\perp$ from the linearized relation, which means that they are related to $h_{\mu\nu}$ nonlinearly.

In summary, we conclude that the background embedding corresponding to the Minkowski space is to be taken in the form of an unfolded embedding the corresponding metric. The Regge–Teitelboim equations are still linearizable on such a background; in the first order, we then than have linearity in corrections to both the embedding function and the metric. The linearized equations have the form

$$(G^{\mu\nu}_{\text{lin}} - \varkappa T^{\mu\nu})\bar{b}^a_{\mu\nu} = 0 \tag{30}$$

(where $G^{\mu\nu}_{\text{lin}}$ is the linearized Einstein tensor, standardly expressed in terms of $h_{\mu\nu}$); they represent six (as a result of the above-mentioned transversality $\bar{b}^a_{\mu\nu}$ with respect to the subscript a) linear combinations of the ten Einstein equations.

Linearized equations (30) were studied in [42] in the case of spherical symmetry. If we restrict ourself to the linear approximation, then the solution involves an arbitrariness corresponding to the choice of the distribution of fictitious matter. This arbitrariness can be limited if the second-order equation is taken into account; under the assumption of a static metric (which physically corresponds, for example, to the final stage of galaxy formation), a nonlinear equation for the linear approximation parameters then arises. As a result, we have shown that the arbitrariness in the choice of the background embedding can be chosen such that the resultant gravitational potential agrees well with the observed DM distribution in galactic halos (neglecting deviations from spherical symmetry for real galaxies).

5. Nonrelativistic limit

As mentioned in Sec. 3, in the embedding theory, fictitious matter can be characterized by the set of currents j^μ_a, which are conserved (in the sense of Eqs. (18) and (21)) as a consequence of the equations of motion. In the limit studied in [46], [47], these vectors become nonrelativistic if the j^μ_a are regarded as four ambient-space vectors labeled by the superscript μ:

$$j^\mu_a = \delta^0_a j^\mu + \delta j^\mu_a, \qquad \delta j^\mu_a \to 0. \tag{31}$$

If the metric becomes flat simultaneously with taking this limit, then it can be shown [47] that in some coordinates and for a finite interval of time x^0, the limit background embedding function has the form

$$\bar{y}^0 = x^0, \qquad \bar{y}^I = \bar{y}^I(x^i) \tag{32}$$

(here and hereafter, $i, k, \ldots = 1, 2, 3$ and $I, K, \ldots = 1, \ldots, 9$), where $\bar{y}^I(x^i)$ is an unfolded embedding of the three-dimensional Euclidean metric into ten-dimensional Euclidean space. Because the three-dimensional metric of the general form has six independent components, such an embedding is parameterized by $9-6=3$ arbitrary functions.

The limit EMT of fictitious matter then becomes

$$\bar{\tau}^{\mu\nu} = \bar{\rho}_\tau \delta^\mu_0 \delta^\nu_0, \tag{33}$$

that is, in the limit taken at times from a finite interval of x^0, this matter is at rest, which clarifies the result that follows by taking the limit in the action (see the end of Sec. 3). The equations of motion require the density of fictitious matter to not change with time (again in a range of finite values of x^0), but it can depend arbitrarily on the spatial coordinates x^i; this dependence is determined by the choice of initial data. In the limit, fictitious matter is dust-like and at rest, and therefore it is nonrelativistic before taking the limit.

The second fundamental form corresponding to embedding function (32) is given by

$$\bar{b}^a_{\mu\nu} = \delta^a_I \delta^i_\mu \delta^k_\nu \bar{b}^I_{ik}, \tag{34}$$

where $\bar{b}^I_{ik}$ is transverse with respect to the superscript I (i.e., $b^I_{ik}\partial_m \bar{y}_I = 0$) and, as a consequence of the assumed unfolded embedding, it has rank 6 if understood as a 9×6 matrix with indices I and (ik). This allows introducing the quantity $\bar{\alpha}^{ik}_a$ that is its inverse in a certain sense; it is uniquely defined by the relations

$$\bar{\alpha}^{ik}_I = \bar{\alpha}^{ki}_I, \qquad \bar{\alpha}^{ik}_I \partial_m y^I = 0, \qquad \bar{\alpha}^{ik}_I \bar{b}^I_{lm} = \frac{1}{2}(\delta^i_l \delta^k_m + \delta^i_m \delta^k_l). \tag{35}$$

For a finite range of x^0, both this quantity and $\bar{y}^I$ are independent of time.

However, when taking the relativistic limit, it is necessary to take the speed of light c to infinity, while it relates x^0 to the nonrelativistic time t by the standard formula $x^0 = ct$. As a result, the finite range of t then corresponds to an unbounded range of x^0, and hence the quantity $\bar{y}^I$ acquires a dependence on the time t. At every time instant, it remains an embedding of the three-dimensional Euclidean metric into nine-dimensional Euclidean space, which means that it undergoes isometric bending as time progresses (we recall that such an embedding is parameterized by three functions, but it is unfortunately impossible to write such a parameterization explicitly). With respect to x^0, we have a nonuniform convergence of the background embedding function to its form in (32): it converges only for each finite interval of x^0. We note that along with $\bar{y}^I$, $\bar{\alpha}^{ik}_I$ also acquires a dependence on t.

If we assume that ordinary matter is dust-like and moves slowly, then it can be described by a density ρ. In the nonrelativistic limit, the equations of motion of the embedding theory then reduce [46], [47] to the Poisson equation for the Newtonian gravitational potential φ,

$$\Delta\varphi = 4\pi G(\rho + \rho_\tau) \tag{36}$$

(where G is Newton's gravitational constant), and to the nonrelativistic equations of motion of fictitious matter,

$$\partial_t \psi = \varphi + \frac{1}{2}\gamma^I \gamma^I, \tag{37}$$

$$\partial_t \bar{y}^I = \gamma^I, \tag{38}$$

$$\partial_t \rho_\tau = -\partial_i(\rho_\tau v^i_\tau), \tag{39}$$

$$\partial_t(\rho_\tau v^m_\tau) = -\rho_\tau \partial_m \varphi + \partial_l \{\rho_\tau \bar{\alpha}^{lm}_I [\bar{\alpha}^{ik}_I ((\partial_i \gamma^L)(\partial_k \gamma^L) + \partial_i \partial_k \varphi) + 2 v^i_\tau \partial_i \gamma^I]\}, \tag{40}$$

where $\partial_t \equiv \partial/\partial t$ and

$$\gamma^I = (\partial_i \psi)\partial_i y^I + \bar{\alpha}^{ik}_I \partial_i \partial_k \psi. \tag{41}$$

In this nonrelativistic limit, fictitious matter is described by its density $\rho_\tau = \tau^{00}$ and velocity $v^i_\tau = c\tau^{0i}/\rho_\tau$ and also by four variables that can be called "geometric": a scalar function ψ and three implicitly present functions parameterizing the embedding $\bar{y}^I$ of the three-dimensional Euclidean metric into nine-dimensional Euclidean space at each instant of time. With (41), it can be verified that Eq. (38) guarantees that as time progresses, the embedding function experiences just an isometric bending. Thus, fictitious matter is described by eight variables, corresponding to four pairs of canonically conjugate variables, as suggested by the canonical analysis at the end of Sec. 2.

System of equations (37)–(40) is naturally divided into two pairs. The first pair, Eqs. (37) and (38), determines the dynamics of four "geometric" variables and does not contain the "physical" variables ρ_τ and v^i_τ. The second pair, Eqs. (39) and (40), determines the dynamics of "physical" variables, but there is no complete separation of variables because the "geometric" variables enter (40) via the quantities $\bar{\alpha}_I^{ik}$ and γ^I. The physical meaning of Eq. (39) is clear: it is the continuity equation for fictitious matter. The physical meaning of Eq. (40) can be clarified if this equation is rewritten in the form

$$\rho_\tau(\partial_t + v_\tau^i \partial_i)v_\tau^m = -\rho_\tau \partial_m \varphi + \partial_l\{\rho_\tau[v_\tau^l v_\tau^m + \bar{\alpha}_I^{lm}(\bar{\alpha}_I^{ik}((\partial_i \gamma^L)(\partial_k \gamma^L) + \partial_i \partial_k \varphi) + 2v_\tau^i \partial_i \gamma^I)]\}. \tag{42}$$

In this expression, the acceleration of an "individual particle" of fictitious matter appears in the left-hand side (we recall that there are no such particles from the fundamental standpoint, but it is convenient to discuss fictitious matter in such terms), and then the right-hand side of the equation is the force (per unit volume) acting on this particle. The first term is the usual gravitational force corresponding to the Newtonian approximation, and the remaining terms are some self-action force of fictitious matter, which depends not only on the "physical" characteristics of this matter (the density ρ_τ and the velocity v_τ^i) but also on its "geometric" characteristics ψ and y^I. If the $1/c$ corrections are taken into account, the embedding function corresponding to the nonrelativistic approximation has the form [46], [47]

$$\begin{aligned} y^0 &= ct + \frac{1}{c}\,\psi(t, x^i) + o\left(\frac{1}{c^2}\right), \\ y^I &= \bar{y}^I(t, x^i) + \frac{1}{c^2}\bar{\alpha}^{Iik}\left(\frac{1}{2}(\partial_i \psi)(\partial_k \psi) - \varphi \delta_{ik}\right) + o\left(\frac{1}{c^2}\right). \end{aligned} \tag{43}$$

In the embedding theory, the presence of the self-action for fictitious matter allows us to hope that in the case of its identification with DM, the "core–cusp" problem mentioned in Sec. 1 can be solved. In solving this problem, the method of analytic evaluation of the matter distribution profile [48] can be used to avoid time-consuming simulations taking deviations from the Newtonian behavior of fictitious matter particles due to self-action into account. The method amounts to considering the particle distribution function along possible trajectories of motion and finding the relation between the behavior of the profile and the asymptotics at zero of the particle distribution function with respect to the absolute value of the angular momentum.

6. Conclusions

The modified theory of gravity proposed by Regge and Teitelboim [22], the embedding theory, has a simple geometric meaning, which amounts to the assumption that our space–time is a four-dimensional surface in a flat ambient space. On the other hand, the embedding theory can be understood as a result of changing the independent variable in the GR action, Eq. (10), which relates this theory to mimetic gravity, a theory popular in recent years [13], and to other modifications of gravity arising as a result of changes of variables involving derivatives. The advantage of the embedding theory over alternative versions is the clear geometric meaning of the formula for the change of variables.

In the weak gravitational field limit, the principle of superposition for gravity requires that an unfolded surface [45] be used as the background embedding (corresponding to the Minkowski space metric). Finding the explicit form of all unfolded embeddings for the Minkowski space metric remains an unsolved mathematical problem.

When reformulating the embedding theory in the GR form with fictitious matter, the emerging fictitious matter has a sufficiently large number of degrees of freedom: there are four of them, which allows arbitrarily setting eight initial functions at the initial time instant. Such a large number of degrees of freedom allows us to hope to successfully explain many effects usually associated with DM; this is in contrast to mimetic gravity, which requires further complicating the theory by introducing new terms to obtain the same result. For the embedding theory, which turns out to be sufficiently complex originally, there is a hope to do this without additional complications; but the complexity of its equations of motion makes it difficult to derive consequences that could be compared with DM observations.

On the scale of galaxies, this problem reduces to analyzing solutions of nonrelativistic equations (37)–(40). It is necessary to understand the properties of the self-action force of fictitious matter and the distribution profile of fictitious matter in galaxies after the inclusion of fictitious matter. Finding the behavior of this profile from the equations in the domain of galactic halos would allow comparing the results with the observed galaxy rotation curves, and finding its behavior at the center of a galaxy would indicates whether the "core–cusp" problem mentioned in Sec. 1 can be solved in the framework of this approach.

It is also necessary to study the behavior of fictitious matter on cosmological scales, under the assumptions of uniformity and isotropy of the distribution of ordinary matter, which are usual for Friedmann models. Such a study was conducted in [40], [41]. It was shown that for some choice of the initial ratio of the amount of fictitious to ordinary matter (the latter containing the cosmological constant), it is possible to obtain the ratio observed at present [40]; however, for a natural ratio taken at the beginning of the inflation epoch, the contribution of fictitious matter turns out to be strongly suppressed at the present time [41].

We emphasize that in both papers sited above, the embedding into a five-dimensional ambient space was chosen as a surface corresponding to the metric of the Friedmann model. If we assume that the ambient space of the embedding theory is ten-dimensional, then in the framework of the symmetry approximation of the Friedmann model, the surface is embedded into a five-dimensional subspace of the ambient space. As a consequence, the corresponding embedding is not an unfolded one. If this is true not only "on average" (on very large scales) but also on small scales, then the principle of superposition for the gravitational field is not satisfied (see Sec. 4), which contradicts the observations. Therefore, we have to assume that only some averaged surface lies in a five-dimensional subspace of the ambient space, and a more precise analysis on small scales shows the exact surface "goes out" of that subspace, i.e., deviations in the transverse directions become noticeable. Because of the nonlinearity of the Regge–Teitelboim equations, it seems unlikely that the results obtained for a precisely five-dimensional embedding would then remain true. It is therefore necessary to repeat the study presented here, either including the extra dimensions into which the surface "goes out" on small scales or originally choosing the embedding for the Friedmann metric that is also unfolded on cosmological scales.

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