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Localized Waves Propagating Along an Angular Junction of Two Thin Semi-Infinite Elastic Membranes Terminating an Acoustic Medium

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Abstract. We study the existence of localized waves that can propagate in an acoustic medium bounded by two thin semi-infinite elastic membranes along their common edge. The membranes terminate an infinite wedge that is filled by the medium, and are rigidly connected at the points of their common edge. The acoustic pressure of the medium in the wedge satisfies the Helmholtz equation and the third-order boundary conditions on the bounding membranes as well as the other appropriate conditions like contact conditions at the edge. The existence of such localized waves is equivalent to existence of the discrete spectrum of a semi-bounded self-adjoint operator attributed to this problem. In order to compute the eigenvalues and eigenfunctions, we make use of an integral representation (of the Sommerfeld type) for the solutions and reduce the problem to functional equations. Their nontrivial solutions from a relevant class of functions exist only for some values of the spectral parameter. The asymptotics of the solutions (eigenfunctions) is also addressed. The far-zone asymptotics contains exponentially vanishing terms. The corresponding solutions exist only for some specific range of physical and geometrical parameters of the problem at hand.

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1. INTRODUCTION

In this section, we formulate the boundary-value problem in a wedge (Fig. 1) as a spectral problem with the aim to specify the discrete spectrum of the corresponding operator. In the statement of the problem, together with the spectral parameter, some other physical and geometrical parameters are present. We also discuss the main result. In this section we comment on the connection of the problem at hand with the physical problem of propagation of localized acoustic waves along the edge of junction of two thin elastic membranes. Then we discuss some known results of similar nature and briefly describe further content of our work.

1.1. Classical statement of the problem for acoustic pressure u

Introduce polar coordinates in the angular domain $\Omega=\Omega_+\cup\Omega_-\cup Ox=\{(r,\varphi):\, r>0,\ |\varphi|<\Phi\}$, (see Fig. 2), $x=r\cos\varphi,\ y=r\sin\varphi,\ \partial\Omega=l_+\cup l_-$. We look for nontrivial classical solutions u that satisfy the equation

$$-\triangle u(r,\varphi;\varkappa)\,=\,-\varkappa^2\,u(r,\varphi;\varkappa),\tag{1}$$

$\triangle=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$. In our case the spectral parameter $-\varkappa^2=k^2-k_e^2$ is assumed to be negative.

The boundary condition on $l_\pm$ reads

$$\mathcal{L}u|_{l_\pm}:=\left\{\frac{\partial^2}{\partial r^2}+k_0^2\right\}\frac{\pm1}{r}\frac{\partial u}{\partial\varphi}-\nu_*u\bigg|_{\varphi=\pm\Phi}=0\,,\tag{2}$$

where $\nu_*=\rho\omega^2\mathcal{N}_*>0,\ k_0^2=k_M^2-k_e^2>0$ are parameters discussed below, the normal n on $l_\pm$ is directed from $\Omega_\pm$ and $\frac{\partial}{\partial n}\big|_{l_\pm}=\pm\frac{1}{r}\frac{\partial}{\partial\varphi}\Big|_{\varphi=\pm\Phi}$.

However, in what follows we shall study only solutions *symmetric* with respect to the axis Ox, $u(r,\varphi;\varkappa)=u(r,-\varphi;\varkappa)$. Thus it is sufficient to find u in Ω_+. As a result, we have

$$\frac{1}{r}\frac{\partial u}{\partial\varphi}\bigg|_{\varphi=0}=0\,,\ r>0.\tag{3}$$

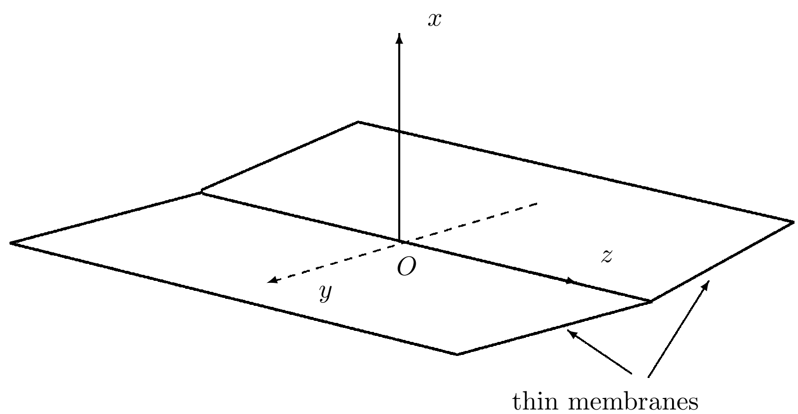

Fig. 1. Angular contact of two thin elastic membranes

It is known that for the formulation of the boundary-value problem for the Helmholtz equation with high order boundary conditions, e.g., with condition (2) on the surface of a thin elastic membrane, it is necessary to postulate supplementary conditions, the so-called contact condition (see also Chapter 8 in [5]). In our case, this is a condition of rigid jamming of the membrane at the point O,

$$\left.\frac{\partial u}{\partial \varphi}\right|_{\varphi=0,\, r=0+} = 0\,. \tag{4}$$

The contact condition (4) implies that the membrane is immovable (pinched) at O so that there is not shift in the direction orthogonal to the line l_+ of the membrane at this point, $\xi_{r=0,\varphi=\Phi} = 0$. The contact condition is necessary for the formal symmetry of the operator attributed to the problem.

We seek classical solutions that satisfy a Meixner's type condition at the angular point

$$u(r,\varphi) = B + \mathrm{O}(r^{\delta}), \;\; r \to 0\,, \;\delta > 0\,, \tag{5}$$

uniformly with respect to φ, B is a constant. Condition (5) implies that $u \in H^1_{loc}(\Omega_+)$.

We expect to determine a discrete set of values $-\varkappa^2$ for which the problem (1)–(5) has a nontrivial solutions (from H^1) exponentially vanishing as $r \to \infty$ and such that

$$\int_0^{\Phi}\int_0^{\infty} |u(r,\varphi;\varkappa)|^2 \exp(2dr)\, r\, \mathrm{d}r\, \mathrm{d}\varphi \,<\, \infty \tag{6}$$

for some $d > 0$.

In what follows we *postulate* the above-mentioned inequalities for k_M, k_e, k and, therefore, appropriate bounds for the physical parameters k_M, k, ν_* and Φ and study existence of the negative eigenvalues $-(\varkappa_m)^2$, $m = 1, 2, \ldots, N_\Phi - 1$ for the problem (1)–(5), N_Φ is defined below.

1.1.1. The main results. In this work we show that edge waves can propagate along the edge Oz and are exponentially localized in orthogonal to the edge direction in an acoustic medium (Fig. 1). They are described by

$$U^m_{\pm}(x,y,z) = \;\mathrm{e}^{\pm\,\mathrm{i}\,k^m_e\, z}\, u(r,\varphi;\varkappa_m)$$

with the wave number $k^m_e = \sqrt{k^2 + \varkappa^2_m}$, where $-(\varkappa^m)^2$ and $u(r,\varphi;\varkappa_m)$, $m = 1, \ldots, N_\Phi - 1$ are eigenvalues and eigenfunctions of a self-adjoint operator attributed to the problem at hand (see Sect. 1.3). It is obvious that $k_M > k^m_e > k$ or $c_M < c^m_e < c$ for the velocities. As we imply, $-(\varkappa_m)^2$ and $u(r,\varphi;\varkappa_m)$ must satisfy problem (1)–(6) in the classical sense.

The precise statement about the existence of the eigenvalues and eigenfunctions is given by Theorem 4.1. It claims that there exists a nonempty set G_* of physical parameters k_M, k, ν_* and Φ of the problem (1)–(6)

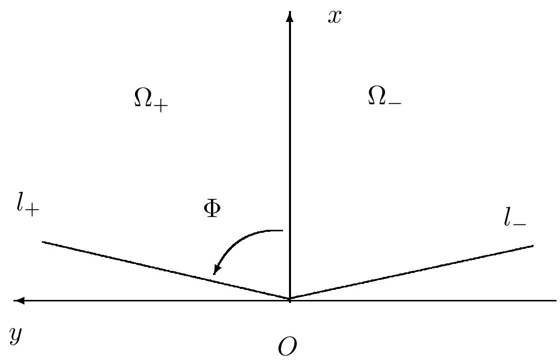

Fig. 2. Angular domain $\overline{\Omega_+ \cup \Omega_-}$

such that $-\varkappa_m^2$ and $u(r,\varphi,\varkappa_m)$, $m = 1, \dots N_\Phi - 1$ are the eigenvalues and eigenfunctions. Here

$$\varkappa_m = \frac{v_0(k_M^2 - k^2, \nu_*)}{\sin \vartheta_0^m},$$

$N_\Phi =$ int $\left(\frac{1}{2}\left[\frac{\pi}{2\Phi} + 1\right]\right)$ and $\vartheta_m = \Phi(2m-1)$, $v_0(k_M^2 - k^2, \nu_*)$ is a root of the algebraic equation

$$v^3 - (k_M^2 - k^2)v + \nu_* = 0,$$

satisfying $0 < v_0 < 1$. The eigenfunctions take the form of the Sommerfeld integral (28)

$$u(r,\varphi;\varkappa_m) = \frac{1}{2\pi\,\mathrm{i}} \int\limits_{\Gamma} \mathrm{d}z\ \mathrm{e}^{\varkappa_m r \cos(z)} f_m(z+\varphi)\,,$$

where f_m is a solution of some functional equation in a special class of meromorphic functions, Γ is shown in Fig. 3. $u(r,\varphi;\varkappa_m)$ exponentially vanishes as $r \to \infty$.

The number of waves $U_\pm^m(x,y,z)$ is finite, which is not proven herein. Such waves really exist provided the wedge's opening 2Φ is less that π. The less the angle of opening the more waves with different wave numbers can propagate along the edge. Contrary to the more sophisticated problems like those in [15, 16], we could show not only existence of such waves but also gave explicit expressions for them in terms of the computed eigenfunctions and eigenvalues.

1.2. Physical motivation and localized edge waves in an acoustic medium

Consider the acoustic pressure $U(x,y,z)$ ($\exp(-\,\mathrm{i}\omega t)$ time dependence is assumed throughout the paper) in the medium over the membranes (Fig. 1). $U(x,y,z)$ satisfies the stationary wave (Helmholtz) equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U(x,y,z) + k^2 U(x,y,z) = 0,$$

where $k = \omega/c$, c is the velocity in the acoustic medium. We look for a solution of the latter equation in the form of waves

$$U_\pm(x,y,z) = \mathrm{e}^{\pm\,\mathrm{i}\,k_e\,z}\, u(r,\varphi;\varkappa) \tag{7}$$

propagating along the axis Oz in both directions, (see Fig. 1), $k_e = \omega/c_e$, c_e is the wave velocity of the edge wave which is yet unknown.

Substitute $U_\pm$ into the Helmholtz equation for and arrive at the equation (1) with

$$\varkappa^2 = k_e^2 - k^2 > 0.$$

The latter inequality implies that the wave velocity of the edge wave is less than the wave velocity in the acoustic medium.

Transverse displacement ζ of a membrane is proportional to the normal derivative of the acoustic pressure on the membranes l_+ and l_-,

$$\zeta(s,z,t) = \frac{\mathrm{e}^{\pm\, \mathrm{i}\, k_e\, z - \,\mathrm{i}\omega t}}{\rho\omega^2}\, \frac{\partial u}{\partial N}\bigg|_{l_\pm} = \mathrm{e}^{\pm\, \mathrm{i}\, k_e\, z -\, \mathrm{i}\omega t}\, \xi(s)$$

with

$$\xi(s) = \frac{1}{\rho\omega^2}\frac{\partial u}{\partial N}\bigg|_{l_\pm},$$

N is the normal directed to the acoustic medium in Ω, ρ is the density of an acoustic medium, s is the coordinate along the membranes, see also [2].

Consider the wave equation for the transverse displacement ζ of a thin elastic membrane under external acoustic pressure acting from Ω onto the membranes,

$$(\triangle_{s,z} - c_M^{-2}\partial_t^2)\zeta = -\frac{\mathrm{e}^{\pm\, \mathrm{i}\, k_e\, z -\, \mathrm{i}\omega t}}{T}\, u|_{\partial\Omega}$$

($T = \mathrm{const}$ is tension per square unit, $\rho = \mathrm{const}$ is the density of the acoustic medium), see also [2], we arrive at the condition (2) that follows from the equation

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2} + k_0^2\right\}\xi(s) = -\mathcal{N}_* u|_{l_\pm},$$

with $\mathcal{N}_* = \frac{1}{T}$, $k_0^2 = k_M^2 - k_e^2$, that is valid on the on the surfaces of the membranes so that condition (2) is satisfied. The wave velocity $c_M = \sqrt{\rho_M/T}$ ($k_M = \omega/c_M$) in an elastic membrane is expected to be less than the yet unknown velocity of the edge wave c_e, $k_e = \omega/c_e$.

The existence of the discrete spectrum means that localized (symmetric) waves (7) with the wave numbers $k_e^m = \sqrt{k^2 + \varkappa_m^2}$ can propagate along the edge of the acoustic wedge terminated by semi-infinite membranes. The energy carried by these waves is concentrated near the axis Oz.

1.3. A comment on a self-adjoint operator attributed to the problem

In this section we briefly describe a way to associate the problem with a self-adjoint operator A, the negative discrete spectrum of which is of interest in our case. We write both the equation for u in (1), which is obtained from the equation for $U_\pm$, and the condition connecting u and ξ on the boundary $\partial\Omega$ in the matrix form

$$\begin{pmatrix} -(\triangle + k^2) & 0 \\ -\mathcal{N}_*(\cdot)|_{l_\pm} & -\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2} + k_M^2\right\}\end{pmatrix}\begin{pmatrix} u \\ \xi \end{pmatrix} = E\begin{pmatrix} u \\ \xi\end{pmatrix},$$

$E = -k_e^2$ is the spectral parameter, $(\cdot)|_{l_\pm}$ is the trace operator that establishes correspondence between u and its value $u|_{l_\pm}$ on the boundary $l_\pm$.

We could consider the sesquilinear (quadratic) form Q_A of the matrix differential operator in the left-hand side of the latter equation. Consider a Hilbert space $\mathcal{H} = L_2(\Omega) \oplus L_2(\partial\Omega)$ and $h = (u,\xi)^t \in \mathcal{H}$. We take h from $\mathrm{Dom}[Q_A] \subset \left\{h \in H^1(\Omega) \oplus H^1(\partial\Omega), \text{ such that } \xi(O) = 0,\ \xi(s) = \frac{1}{\rho\omega^2}\frac{\partial u}{\partial N}\Big|_{\partial\Omega}\right\}$ then we define the form Q_A in a traditional manner from the explicit expression of the matrix differential operator in the equation above.

This form is semi-bounded and densely defined. Provided it is closable, which should be verified, it uniquely specifies a self-adjoint semi-bounded operator $A = A^*$ in $\mathcal{H}$. Namely this operator should be attributed to the problem at hand. We are interested in the negative discrete spectrum of this operator if the latter is not empty. A traditional way to study spectrum of such operators is based, in particular, on the variational principle or (and) separation of variables (see, e.g. [7, 1, 3]). Actually the spectrum of $A = A^*$ consists of two components: essential spectrum $\sigma_{ess} = [E_*, \infty)$, ($E_* < 0$) and a (finite) discrete set σ_d consisting of a number of negative eigenvalues $\{E_m\}$ with the eigenfunctions $\{h_m\}$.[1]

However, we follow an alternative way. In our case the model is explicitly solvable and we constructively determine eigenvalues and eigenfunctions. The completeness of the constructed eigenvalues is not considered, nevertheless, we think that we have constructed all of them. We also connect this eigenvalues and eigenfunction with the localized waves propagating along the common edge of two contacting membranes terminating an acoustic medium.

[1]These facts need to be proved.

1.4. Some comments on the literature and on the content of the work

The study of the propagating edge waves has its long lasting history. The numerical and experimental study supplemented by physical argumentation has been carried out in a number of works, e.g., [8–12]. However, the mathematical proof of the existence of such localized solutions has been given later, see [13–15], where the existence of such localized waves and some estimates for solutions were linked to study of the discrete component of the spectrum of a self-adjoint operator attributed to the problem. An asymptotic approach for an elastic wedge is developed in [16], where a satisfactory survey on the subject is also given.

In our recent works [17–19], incomplete separation of variables and use of the appropriate integral transforms reduces the problem to determine localized solutions in wedge- or cone-shaped domains to study the spectral properties of some functional-difference equations, see also [4] in this context.

These equations are then reduced to integral equations with the operator represented as a self-adjoint compact perturbation of the so called Mehler operator [20], the spectral analysis of which is given in explicit terms. In particular, the existence of the discrete spectrum is studied.

In this work we seek solutions of problem (1)–(5) in the form of the Sommerfeld integral with the unknown meromorphic function in the integrand (Sommerfeld transformant). The integral solves equation (1). The substitution into the boundary conditions (2), (3) leads to a system of Malyuzhinets functional equations. The functional equations are solved in a class of meromorphic functions. However, appropriate solutions exist if a parameter in the equations takes values from a discrete set. These values directly specify the desired values of $\varkappa_m$ that are responsible for existence of the localized solutions. As a result, for every particular m existence of such solutions is connected with solutions of an algebraic equation $v^3 - (k_M^2 - k^2)v + \nu_* = 0$, depending on physical and geometrical characteristics of the problem at hand. We study roots of this equation for a domain of physical parameters and identify a sub-domain, where such localized solutions exist. This leads to construction of the corresponding eigenvalues and of the eigenfunctions in the form of the Sommerfeld integral. The asymptotics of an eigenfunction, as $r \to \infty$, is obtained by means of the saddle point technique. We find that the eigenfunction exponentially vanish at infinity, so that (6) is valid, satisfies the Meixner's type condition (5) as well as the contact condition (4).

2. REDUCTION OF THE PROBLEM TO THAT FOR THE FUNCTIONAL EQUATIONS IN A CLASS OF MEROMORPHIC FUNCTIONS

One of the natural ideas is to make use of the Sommerfeld integral representation for the solution of the problem in Ω_+,

$$u(r, \varphi; \varkappa) = \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma} \mathrm{d}z\;\; \mathrm{e}^{\varkappa r \cos(z)}\, f(z + \varphi), \tag{8}$$

f is a meromorphic function (Sommerfeld transformant) depending on $\varkappa > 0$ and other parameters, however, this dependence is omitted for compactness. The integration contour Γ is shown in Fig. 3, $\Gamma_+ = (\,\mathrm{i}\infty + \pi,\ \mathrm{i}b_0 + \pi] \cup [\,\mathrm{i}b_0 + \pi,\ \mathrm{i}b_0 - \pi] \cup [\,\mathrm{i}b_0 - \pi, \pi +\ \mathrm{i}\infty)$, $b_0 > 0$ and Γ_- is symmetric to Γ_+ with respect to the origin O.

The class of Sommerfeld transformants f is described below. The integral rapidly converges so that the calculations below are easily justified.

In order to substitute (8) into the equation and boundary conditions and derive the functional equations for f we, first, describe a class of functions. We assume that f is meromorphic having its poles in some bounded strip $|\Im(z)| < b_0$, it is holomorphic in $\Pi(-a, a) = \{z :\ -a < \Re(z) < a\}$ for some $a > 0$ and have finite limits $f(\pm\,\mathrm{i}\infty)$ with $f(\,\mathrm{i}\infty) = -f(-\,\mathrm{i}\infty)$. We also require

$$|f(z) - f(\pm\,\mathrm{i}\infty)| \leqslant C \exp(-\delta|z|) \tag{9}$$

as $z \to \pm\,\mathrm{i}\infty$ in $\Pi(-a, a)$ for some $\delta > 0$. This class of such meromorphic functions is denoted by $\mathcal{M}$. Condition (9) enables one to ensure the Meixner's condition (5).

By the direct substitution we verify that ($\zeta = \varkappa r$)

$$\left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\zeta\frac{\partial}{\partial\zeta} + \frac{1}{\zeta^2}\frac{\partial^2}{\partial\varphi^2} - 1\right) u(r, \varphi; \varkappa) = \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma} \mathrm{d}z\;\; \mathrm{e}^{\zeta r \cos(z)} (\cos^2 z + \sin^2 z - 1) f(z + \varphi) = 0\,,$$

where we used that $\frac{\mathrm{d}}{\mathrm{d}\varphi} f(z + \varphi) = \frac{\mathrm{d}}{\mathrm{d}z} f(z + \varphi)$ and integrated by parts. Remark that, due to the exponent in the integrand which rapidly vanishes at the ends of Γ, the integral rapidly converges.

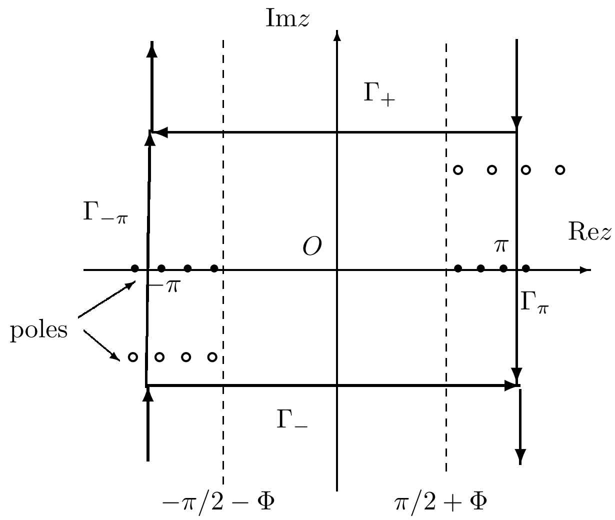

Fig. 3. The contour $\Gamma = \Gamma_+ \cup \Gamma_-$ and some sigularities of $f(z)$

Now we turn to the boundary condition (2),

$$\frac{1}{\varkappa^3}\mathcal{L}u|_{l_+} := \frac{1}{2\pi\,\mathrm{i}}\int\limits_{\Gamma} \mathrm{d}z\;\; \mathrm{e}^{\varkappa r\cos(z)}\left([\cos^2 z + k_0^2/\varkappa^2]\frac{f_z'(z+\Phi)}{\varkappa r} - \frac{\nu_*}{\varkappa^3}f(z+\Phi)\right)$$
$$= \frac{1}{2\pi\,\mathrm{i}}\int\limits_{\Gamma} \mathrm{d}z\;\; \mathrm{e}^{\varkappa r\cos(z)}\left([\cos^2 z + k_0^2/\varkappa^2]\sin(z)\,f(z+\Phi) - \frac{\nu_*}{\varkappa^3}f(z+\Phi)\right) = 0,$$

In this equality, traditionally, we make use of the Malyuzhinets theorem [21, ?] then arrive at the functional equation

$$[\sin^3 z - (1+D)\sin z + \nu]f(z+\Phi) - [\sin^3(-z) + (1+D)\sin(-z) + \nu]f(-z+\Phi) = \sin z(c_0 + c_1\cos z)), \quad (10)$$

where we introduced notations

$$D = \frac{k_0^2}{\varkappa^2}, \qquad \nu = \frac{\nu_*}{\varkappa^3}$$

and

$$1 + D = 1 + \frac{k_M^2 - k_e^2}{k_e^2 - k^2} = \frac{k_M^2 - k^2}{\varkappa^2}.$$

The constants c_0, c_1 are still unknown and will be chosen later. Equation (10) is to be solved in the introduced class of meromorphic functions.

In the same manner, from the boundary condition (3) we find that

$$f(z) + f(-z) = 0, \quad (11)$$

which means that the Sommerfeld transformant f is odd. In view of (11) and symmetry of the integration contour we can write (8) in the form

$$u(r,\varphi;\varkappa) = \frac{1}{2\pi\,\mathrm{i}}\int\limits_{\Gamma} \mathrm{d}z\;\; \mathrm{e}^{\varkappa r\cos(z)}\frac{1}{2}[f(z+\varphi) + f(z-\varphi)] \quad (12)$$

so that $u(r,\varphi;\varkappa) = u(r,-\varphi;\varkappa)$ is obvious.

Remark 1. Our basic goal is to find odd solutions of the functional equation (10) in such a form that the solutions $u(r,\varphi;\varkappa)$ in (12) rapidly vanish as $r\to\infty$ so that the condition (6) is valid. As we can see below, this can be achieved only for some particular discrete values $\varkappa = \varkappa_m$. The corresponding $-\varkappa_m^2$ and $u(r,\varphi;\varkappa_m)$ will be the desired eigenvalues and eigenfunctions.

Let $w := \sin z$. We consider roots of the polynomial (see (10))

$$w^3 - (1+D)w + \nu = (w - w_0)(w - w_1)(w - w_2) = 0\,.$$

Now we make an important assumption about the roots $w_j,\ j = 0,1,2$, namely, let the following inequalities hold[2]

$$0 < w_0 < 1,\ \ w_1 > 1,\ \ w_2 < -1\,. \tag{13}$$

The prescribed bounds (13) will be commented below and are specified by appropriate values of the coefficients $1+D$ and ν for the polynomial. In conditions (13) we can parametrize the roots as follows,

$$w_0 = \sin\vartheta_0,\ \ \vartheta_0 \in (0,\pi/2),$$

$$w_1 = \sin\vartheta_1,\ \ \vartheta_1 = \pi/2 + \mathrm{i}\tau_1,\ \ \tau_1 > 0,\ \ w_2 = \sin\vartheta_2,\ \ \vartheta_2 = -\pi/2 + \mathrm{i}\tau_2,\ \ \tau_2 > 0,$$

where $\vartheta_j,\ j = 0,1,2$ are actually defined by the coefficients $1+D$ and ν. The roots of a polynomial of the third degree are given explicitly by known algebraic expressions in terms of its coefficients.[3] Taking into account the introduced notations, we write the equation (10) as follows

$$\prod_{j=0}^{2}[\sin z - \sin\vartheta_j]f(z+\Phi) - \prod_{j=0}^{2}[-\sin z - \sin\vartheta_j]f(-z+\Phi) = \sin z(c_0 + c_1\cos z). \tag{14}$$

The main argumentation to find the eigenfunctions and eigenvalues is as follows. We can construct solutions of (14) from $\mathcal{M}$ and then determine the Sommerfeld integral representation for $u(r,\varphi;\varkappa)$ in such a form that it has exponential decay as $r\to\infty$ in order to satisfy condition (6). To this end, we have to find the transformant f from $\mathcal{M}$ with additional requirements which ensure the desired exponential decay. These requirements imply that the transformant f has no poles in the closed strip $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$, see Fig. 3, where the poles (circles) are located outside this closed strip. Indeed, in order to compute the asymptotics of the Sommerfeld, as $r\to\infty$, one has to deform the Sommerfeld double-loop contour $\Gamma = \Gamma_+\cup\Gamma_-$ into the steepest descent paths (SDPs) Γ_π and $\Gamma_{-\pi}$ in Fig. 3 going across the saddle point $+\pi$ and $-\pi$ correspondingly. In the process of such deformation no poles from the closed strip $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$ (i.e. also on the boundary) must be captured, otherwise, the terms of the corresponding residue contributions either exponentially grow or, at most, are bounded as $r\to\infty$. As a result, we have to construct the transformant f that is regular (holomorphic) in $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$ and has no poles on its boundary. It is crucial that such transformant exists only for a finite set of values $\vartheta_0 = \vartheta_0^s,\ \ s = 1,2\ldots N_\Phi - 1$.

3. MEROMORPHIC SOLUTIONS $f\in\mathcal{M}$ OF THE FUNCTIONAL EQUATIONS THAT ARE REGULAR IN THE CLOSED STRIP $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$

We are looking for the desired meromorphic solutions in the form

$$f(z) = \frac{f_0(z)}{\sin(\mu z)}F(z)S(z), \tag{15}$$

where $\mu = \pi/(2\Phi)$, f_0 is even and satisfies an auxiliary equation

$$[\sin z - \sin\vartheta_0]f_0(z+\Phi) - [-\sin z - \sin\vartheta_0]f_0(-z+\Phi) = 0. \tag{16}$$

F is even and solves a similar equation

$$\prod_{j=1}^{2}[\sin z - \sin\vartheta_j]F(z+\Phi) - \prod_{j=1}^{2}[-\sin z - \sin\vartheta_j]F(-z+\Phi) = 0\,. \tag{17}$$

The following Lemma is a direct consequence of (14)–(17) and is verified by a simple substitution.

[2]These assumptions will enable one to construct the desired eigenfunctions and eigenvalues for a range of physical parameters in the problem at hand.

[3]The corresponding explicit Cardano formulas are well known but rather inefficient for our needs.

Lemma 3.1. *Let an even meromorphic function $S(\cdot)$ solve the inhomogeneous functional equations*

$$S(z+\Phi)-S(-z+\Phi)=\chi(z)\,,$$
$$S(z-\Phi)-S(-z-\Phi)=-\chi(z) \tag{18}$$

with

$$\chi(z)=\frac{\sin z(c_0+c_1\cos z)\sin(\mu[z+\Phi])}{\prod_{j=0}^{2}[\sin z-\sin\vartheta_j]f_0(z+\Phi)F(z+\Phi)}\,.$$

Then f defined by (15) is an odd meromorphic solution of equations (14).

3.1. Solution of the equation for f_0 and an analysis

Equation (16) can be solved explicitly in terms of the Malyuzhints function ψ_Φ, see [19],[21], Chapter 1,

$$f_0(z)=\psi_{\Phi/2}(z+\pi/2+\vartheta_0)\psi_{\Phi/2}(z-\pi/2-\vartheta_0).$$

f_0 has no poles in the strip $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$ and on its boundary.

Remark 2. Recall that Malyuzhinets function is a 'minimally' growing at $\pm\,\mathrm{i}\infty$ solution of the functional equation

$$\frac{\psi_\Phi(z+2\Phi)}{\psi_\Phi(z-2\Phi)}=\cot\left(\frac{z}{2}+\frac{\pi}{4}\right),$$

having no poles and zeroes in the strip $\Pi(-\pi/2-2\Phi,\pi/2+2\Phi)$ and in this strip is represented by the expression

$$\psi_\Phi(z)=\exp\left\{-\frac{1}{2}\int_0^\infty \mathrm{d}t\,\frac{\cosh(zt)-1}{t\cos(\pi t/2)\sin(2\Phi t)}\right\}$$

with the asymptotics $\psi_\Phi(z)=C\exp(\mp\,\mathrm{i}\mu z/4)[1+o(1)]$ as $z\to\pm\,\mathrm{i}\infty$, $\psi_\Phi(z)=\psi_\Phi(-z)$.

The zeroes of the Malyuzhinets function are located at the points

$$\zeta^\pm_{jn}=\pm\left(\frac{\pi}{2}[4j-3]+2\Phi[2n-1]\right),\ \ j,n=1,2,3\ldots,$$

whereas the poles are at

$$\xi^\pm_{jn}=\pm\left(\frac{\pi}{2}[4j-1]+2\Phi[2n-1]\right),\ \ j,n=1,2,3\ldots.$$

The auxiliary function f_0 is of $O(\cos\mu z)$ as $z\to\pm\,\mathrm{i}\infty$ in the strip $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$. In order to compensate such its growth at infinity, we divide it by $\sin\mu z$ so that $\frac{f_0(z)}{\sin(\mu z)}$ solves equation (16), is odd and is bounded at infinity in the strip $\Pi(-\pi/2-\Phi,\pi/2+\Phi)$.[4]

However, such a trick contributes a number of additional poles at zeroes of $\sin\mu z$. Indeed, the poles at $z=2\Phi n,\ \ n=0,\pm1,\pm2\ldots$, that are located in the closed strip $\Pi[-\pi/2-2\Phi,\pi/2+2\Phi]$, can be captured in the process of deformation of the Sommerfeld contour Γ into steepest descent (SD) paths $\Gamma_\pi\cup\Gamma_{-\pi}$, Fig. 3. As a result, the corresponding residue contributions would lead to exponentially growing (or only bounded) terms in the asymptotics of the solution. To get exponential decay of the solution $u(r,\varphi;\varkappa)$ we choose the parameter ϑ_0 in the auxiliary function f_0 appropriately in order to compensate the poles at $z=2\Phi n,\ \ n=0,\pm1,\pm2\ldots$ by zeroes of $f_0(z)=\psi_{\Phi/2}(z+\pi/2+\vartheta_0)\psi_{\Phi/2}(z-\pi/2-\vartheta_0)$.

Taking into account the zeroes of the Malyuzhinets functions $\psi_{\Phi/2}$, we find that f_0 has zeroes at the points

$$\{\Phi-\vartheta_0,3\Phi-\vartheta_0,5\Phi-\vartheta_0,\ldots\}$$

and, due to the evenness of f_0, at the symmetric points

$$\{-\Phi+\vartheta_0,-3\Phi+\vartheta_0,-5\Phi+\vartheta_0,\ldots\}\,.$$

The zeroes of $\sin\mu z$ are at $z=2\Phi m,\ \ m=0,\pm1\ldots$, the set of zeroes is

$$\{\cdots-4\Phi,-2\Phi,0,\,2\Phi,\,4\Phi,\ldots\}\,.$$

[4]The boundedness is necessary for the Meixner's condition (5).

The numbers m for which these zeroes are located in the forbidden strip satisfy the inequality

$$-\frac{\pi}{2} - \Phi \leqslant 2\Phi m \leqslant \frac{\pi}{2} + \Phi.$$

Thus $-\frac{\pi}{2} \leqslant (2m+1)\Phi$ and $\frac{\pi}{2} \geqslant (2m-1)\Phi$. We also conclude

$$\frac{\pi}{2}\frac{1}{(2m+1)} \leqslant \Phi < \frac{\pi}{2}\frac{1}{(2m-1)}, \quad m = 1, 2, \dots .$$

The latter bounds, in particular, mean that, provided $\frac{\pi}{6} \leqslant \Phi < \frac{\pi}{2}$, only one additional pole exists, and we take ϑ_0 from the set

$$\vartheta_0 \in \{\Phi\}$$

The parameter $\vartheta_0 = \Phi$ takes only one value, the number of elements in the set is $M(\Phi) = 1,\ \Phi < \pi/2$. If $\frac{\pi}{10} \leqslant \Phi < \frac{\pi}{6}$, there are three such values

$$\vartheta_0 \in \{-3\Phi, \Phi, 3\Phi\},$$

$M(\Phi) = 3$ and so on. If

$$\Phi \in \left[\frac{\pi}{2}\frac{1}{(2m+1)}, \frac{\pi}{2}\frac{1}{(2m-1)}\right)$$

we take $\vartheta_0 = \vartheta_0^m$,

$$\vartheta_0^m := \Phi(2m-1)$$

with $m = 1, 2, \dots N_\Phi - 1,\quad N_\Phi = \ \mathrm{int}\left(\frac{1}{2}\left[\frac{\pi}{2\Phi} + 1\right]\right)$ (integer part of a number). The number of the additional poles (i.e. of zeroes of $\sin\mu z$) is $M(\Phi) = 2(N_\Phi - 1) - 1$. We obtain

$$\frac{f_0^m(z)}{\sin(\mu z)},$$

$m = 1, 2, \dots N_\Phi - 1$, which are the desired auxiliary solutions of equation (16) and

$$f_0^m(z) = \psi_{\Phi/2}(z + \pi/2 + \vartheta_0^m)\psi_{\Phi/2}(z - \pi/2 - \vartheta_0^m).$$

Remark 3. It is worth noticing that, in view of the correspondence given by the algebraic equation

$$w_0(\varkappa) = \sin\vartheta_0^m,\ \ \vartheta_0^m = \Phi(2m-1),\ m = 1, 2, \dots N_\Phi - 1, \tag{19}$$

where w_0 is a root of the polynomial $w^3 - (1 + D)w + \nu = 0$, such that $0 < w_0 < 1$, see (10), precisely this discrete set of values is responsible for the set of eigenvalues $-\varkappa_m^2$. Provided for some range of parameters Φ, k_M, k, ν_* equation (19) has a solution $\varkappa = \varkappa_m$, then, as we show below, there exists the corresponding eigenfunction $u(r, \Phi; \varkappa_m)$ exponentially vanishing as $r \to \infty$.

3.2. Construction of the auxiliary meromorphic solution F of equation (17)

Our goal in this section is to obtain a meromorphic solution F of equation (17) that has no poles and zeroes in the closed strip $\Pi(-\pi/2 - \Phi, \pi/2 + \Phi)$ and is bounded there at $\pm\,\mathrm{i}\infty$. To this end, we consider $g(z) = \log F(z)$ which is an even holomorphic function in this strip and we obtain from (17)

$$g(z + \Phi) - g(z - \Phi) = \log(R(z; \vartheta_1)R(z; \vartheta_2)), \tag{20}$$

where

$$R(z, \vartheta) = \frac{\sin z + \sin\vartheta}{\sin z - \sin\vartheta}.$$

In order to define a holomorphic branch of $\log(R(z; \vartheta_1)R(z; \vartheta_2))$ in the strip $z \in \Pi(-\pi/2, \pi/2)$, one should conduct branch-cuts from zeroes and poles[5] of $R(z; \vartheta_1)R(z; \vartheta_2)$ i.e from $\vartheta_1, -\vartheta_1, \pi - \vartheta_1, -\pi + \vartheta_1$ and $\vartheta_2, -\vartheta_2, \pi + \vartheta_2, -\pi - \vartheta_2$ at infinity at $\pm\infty$ parallel to the real axis and having no intersection with the strip $\Pi(-\pi/2, \pi/2)$. We also fix the branch by the condition $\log(R(z; \vartheta_1)R(z; \vartheta_2))|_{z=0} = 0$.

[5]These points are located on the boundary of the strip $\Pi(-\pi/2, \pi/2)$.

The right-hand side of equation (20) exponentially vanishes along the imaginary axis as $O(1/\sin(z))$. We make use of the Fourier transform along the imaginary axis, we find

$$g(z) = -\frac{v.p.}{2\pi} \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{-\mathrm{i}zt} G(t)\, \mathrm{d}t\,, \quad G(t) = \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{\mathrm{i}z\zeta} g(\zeta)\, \mathrm{d}\zeta\,.$$

In the same manner, we get

$$\log(R(z;\vartheta_1)R(z;\vartheta_2)) = -\frac{v.p.}{2\pi} \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{-\mathrm{i}zt} r_0(t)\, \mathrm{d}t\,,$$

and

$$r_0(t) = \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{\mathrm{i}z\zeta} \log(R(\zeta;\vartheta_1)R(\zeta;\vartheta_2))\, \mathrm{d}\zeta\,.$$

From equation (20) we obtain

$$-\frac{v.p.}{2\pi} \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{-\mathrm{i}zt} G(t)[\,\mathrm{e}^{-\mathrm{i}\Phi t} - \mathrm{e}^{\mathrm{i}\Phi t}]\, \mathrm{d}t = -\frac{v.p.}{2\pi} \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{-\mathrm{i}zt} r_0(t)\, \mathrm{d}t,$$

then making use of the inverse Fourier transform

$$G(t) = \frac{\mathrm{i}}{2} \frac{r_0(t)}{\sin(\Phi t)}.$$

The desired solution g is recovered as follows

$$g(z) = -\frac{v.p.}{2\pi} \int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{-\mathrm{i}zt} \frac{\mathrm{i}}{2} \frac{r_0(t)}{\sin(\Phi t)}\, \mathrm{d}t = -\,\mathrm{i}\frac{v.p.}{4\pi} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}t \frac{\mathrm{e}^{-\mathrm{i}zt}}{\sin(\Phi t)} \left(\int_{\mathrm{i}\mathbb{R}} \mathrm{e}^{\mathrm{i}t\tau} \log(R(\tau;\vartheta_1)R(\tau;\vartheta_2))\, \mathrm{d}\tau \right).$$

Changing the order of integrations, we obtain ($z \in \Pi(-\Phi, \Phi)$)

$$\begin{aligned} g(z) &= -\,\mathrm{i}\frac{v.p.}{4\pi} \int_{\mathrm{i}\mathbb{R}} \log(R(\tau;\vartheta_1)R(\tau;\vartheta_2))\, \mathrm{d}\tau \left(\int_{\mathrm{i}\mathbb{R}} \mathrm{d}t\, \frac{\mathrm{i}\sin(t[\tau - z])}{\sin(\Phi t)} \right) \\ &= \frac{1}{4\Phi} \int_{\mathrm{i}\mathbb{R}} \log(R(\tau;\vartheta_1)R(\tau;\vartheta_2))\, \tan(\mu[\tau - z])\, \mathrm{d}\tau \\ &= \frac{1}{4\Phi} \int_{\mathrm{i}\mathbb{R}} \log(R(\tau;\vartheta_1)R(\tau;\vartheta_2))\, \frac{1}{2}[\tan(\mu[\tau - z]) - \tan(\mu[-\tau - z])]\, \mathrm{d}\tau\,, \end{aligned}$$

where we exploited formula 3.981(1) from [6]. Finally, we find

$$g(z) = \frac{1}{4\Phi} \int_{\mathrm{i}\mathbb{R}} \frac{\mathrm{d}\tau\, \sin(2\mu\tau)}{\cos(2\mu\tau) + \cos(2\mu z)}\, \log(R(\tau;\vartheta_1)R(\tau;\vartheta_2)),$$

where $z \in \Pi(-\Phi, \Phi)$, $\mu = \frac{\pi}{2\Phi}$. It is worth noticing that g is holomorphic and bounded in the strip $\Pi(-\Phi, \Phi)$. In fact, it is holomorphic in a wider strip $\Pi(-\Phi - \pi/2, \pi/2 + \Phi)$. Indeed, we can deform the integration contour in the representation for g to the right, $\mathrm{i}\mathbb{R} \to \mathrm{i}\mathbb{R} + h$ with $0 < h < \pi/2$. Notice that $\log(\cdot)$ in the integrand is holomorphic in $\Pi(-\pi/2, \pi/2)$. The regularity strip of g is then $\Pi(-\Phi + h, \Phi + h)$. Using the principle of analytic continuation, we observe that g is holomorphic in $\Pi(-\Phi, \pi/2 + \Phi)$ and, due to parity, in $\Pi(-\Phi - \pi/2, \pi/2 + \Phi)$.

We find that the sought auxiliary solution F takes the form

$$F(z) = \exp\left\{ \frac{1}{4\Phi} \int_{\mathrm{i}\mathbb{R}} \frac{\mathrm{d}\tau\, \sin(2\mu\tau)}{\cos(2\mu\tau) + \cos(2\mu z)}\, \log(R(\tau;\vartheta_1)R(\tau;\vartheta_2)) \right\}, \tag{21}$$

is even holomorphic bounded function in the strip $\Pi(-\Phi-\pi/2,\pi/2+\Phi)$. Analytic continuation of F in (21) from the regularity strip onto the right-hand side on complex plane is performed by means of the functional equation (17) written in the form

$$F(\zeta)=R(\zeta-\Phi;\vartheta_1)R(\zeta-\Phi;\vartheta_2)F(\zeta-2\Phi)\,.$$

When $\zeta\in\Pi(\pi/2+\Phi,\pi/2+3\Phi)$ in the left-hand-side, the argument of F in the right-hand side of the latter equation is from $\Pi(-\Phi+\pi/2,\pi/2+\Phi)$, where F is holomorphic. All singularities are specified by the coefficient $R(\zeta-\Phi;\vartheta_1)R(\zeta-\Phi;\vartheta_2)$. Then the equation and parity enable one to determine the values of F in the broader strip $\Pi(-3\Phi-\pi/2,\pi/2+3\Phi)$. In particular, we find that the nearest to the imaginary axis poles of F are located on the lines $\Re(z)=\pm\pi/2\pm\Phi$, at the points

$$\zeta_1=\vartheta_1+\Phi=\frac{\pi}{2}+\,\mathrm{i}\tau_1+\Phi,\quad -\zeta_1=-\vartheta_1-\Phi=-\frac{\pi}{2}-\,\mathrm{i}\tau_1-\Phi,$$

where $\vartheta_1+\Phi$ is a root of the denominator of $R(\zeta-\Phi;\vartheta_1)$. In this way, analytic continuation is carried out onto the whole complex plane.

Remark 4. In accordance with the Remark 3, we must choose $\vartheta_0=\vartheta_0^m$, which leads to the appropriate choice of $\varkappa=\varkappa_m$ and $\vartheta_j=\vartheta_j^m$, with $\sin\vartheta_j^m=w_j(\varkappa_m),\ j=1,2$.

3.3. Solution of equation (18) and choice of the constants c_0, c_1

A particular solution of equation (18) in the desired class of the meromorphic functions can be found in a similar way as in the previous section. An equivalent way is the use of the so called $S-$integral (see [21], Chapter 1), we have

$$S_1(z)=\frac{\mathrm{i}}{8\Phi}\int\limits_{\mathrm{i}\mathbb{R}}\frac{d\tau\,\chi(\tau)\sin(\mu\tau)}{\cos\mu\tau-\sin\mu z}-\frac{\mathrm{i}}{8\Phi}\int\limits_{\mathrm{i}\mathbb{R}}\frac{d\tau\,(-\chi(\tau))\sin(\mu\tau)}{\cos\mu\tau+\sin\mu z}$$

and then general solution S is

$$S(z)=c+S_1(z)=c+\frac{\mathrm{i}}{4\Phi}\int\limits_{\mathrm{i}\mathbb{R}}\frac{d\tau\,\chi(\tau)\sin(2\mu\tau)}{\cos(2\mu\tau)+\cos(2\mu z)}\,, \tag{22}$$

where $z\in\Pi(-\Phi,\Phi)$ and a constant c is a solution of the homogeneous equations. This functions admits a meromorphic continuation on the complex plane. In accordance with the expression for χ in (22) it depends on two other arbitrary constants c_0 and c_1, see (18). We appropriately choose them in this section.

As we remarked, the transformant f must have no poles in the closed strip $\Pi(-\Phi+\pi/2,\pi/2+\Phi)$. To this end, the auxiliary solution S must have no poles in the strip $\Pi(-\Phi+\pi/2,\pi/2+\Phi)$. To ensure this, we shall proceed similarly to the continuation of F in the previous section, i.e. we deform the integration contour $\mathrm{i}\mathbb{R}$ to the right-hand side. In the process of such deformation in the strip $\tau\in\Pi(0,\pi/2)$ we can encounter with only one pole $\tau=\vartheta_0$ of the denominator of χ in the integrand of

$$S_1(z)=\frac{\mathrm{i}}{4\Phi}\int\limits_{\mathrm{i}\mathbb{R}}\frac{d\tau\,\sin(2\mu\tau)}{\cos(2\mu\tau)+\cos(2\mu z)}\frac{\sin\tau(c_0+c_1\cos\tau)\sin(\mu[\tau+\Phi])}{\prod_{j=0}^{2}[\sin\tau-\sin\vartheta_j]f_0(\tau+\Phi)F(\tau+\Phi)}\,.$$

We compensate it by zero of the numerator choosing c_0 and c_1 such that

$$c_0+c_1\cos\vartheta_0=0\,. \tag{23}$$

The solution S is continued as a holomorphic function onto the strip $\Pi(-\Phi+\pi/2,\pi/2+\Phi)$.

Now we obtain the Sommerfeld transformant f in the form

$$f(z)=\frac{f_0(z)}{\sin(\mu z)}F(z)S(z) \tag{24}$$

$$=\frac{f_0(z)}{\sin(\mu z)}F(z)\left(c+\frac{\mathrm{i}}{4\Phi}\int\limits_{\mathrm{i}\mathbb{R}}\frac{d\tau\,\sin(2\mu\tau)}{\cos(2\mu\tau)+\cos(2\mu z)}\frac{\sin\tau(c_0+c_1\cos\tau)\sin(\mu[\tau+\Phi])}{\prod_{j=0}^{2}[\sin\tau-\sin\vartheta_j]f_0(\tau+\Phi)F(\tau+\Phi)}\right)\,.$$

In accordance with our procedure (recall that we have chosen $\vartheta_j=\vartheta_j^m$, for some $m,\ j=0,1,2.$) the transformant is holomorphic in the strip $\Pi(-\Phi+\pi/2,\pi/2+\Phi)$. However, it may have and actually has two poles on the boundary of this strip at the points $z = \pm(\Phi + \vartheta_1)$, $\vartheta_1 = \pi/2 + \mathrm{i}\tau_1$. Indeed, consider the functional equation (14) written as

$$f(z) = \frac{\prod_{j=0}^{2}[\sin(z-\Phi)+\sin\vartheta_j]}{\prod_{j=0}^{2}[\sin(z-\Phi)-\sin\vartheta_j]} f(z-2\Phi) + \frac{\sin(z-\Phi)(c_0 + c_1\cos(z-\Phi))}{\prod_{j=0}^{2}[\sin(z-\Phi)-\sin\vartheta_j]}.$$

Provided $z = \Phi + \vartheta_1$, the right-hand side has a pole at this point due to the corresponding zero of the denominator. We also notice that $f(\cdot - 2\Phi)$ is regular near this point. The pole at the symmetric point is due the oddness. There is no other singularities on the boundary of the strip $\Pi(-\Phi + \pi/2, \pi/2 + \Phi)$. In order to compensate the pole by a zero of the numerator we take

$$\prod_{j=0}^{2}[\sin(z-\Phi)+\sin\vartheta_j] f(z-2\Phi) + \sin(z-\Phi)(c_0 + c_1\cos(z-\Phi))\Big|_{z=\Phi+\vartheta_1} = 0,$$

and

$$\prod_{j=0}^{2}[\sin\vartheta_1+\sin\vartheta_j] f(\vartheta_1-\Phi) + \sin\vartheta_1(c_0 + c_1\cos\vartheta_1) = 0, \tag{25}$$

where the function $f(z-2\Phi)|_{z=\Phi+\vartheta_1} = f(\vartheta_1 - \Phi)$ also depends on c, c_0, c_1 in accordance with (24), $\vartheta_1 - \Phi \in \Pi(-\Phi + \pi/2, \pi/2 + \Phi)$.

From the linear algebraic equations (23) and (25), we could express, for instance, the constants c_0, c_1 in terms of c and then substitute them into the right-hand side of (24).[6] We arrive at the main result of this section

Lemma 3.2. *Let $\vartheta_0 = \vartheta_0^m$, $m = 1, 2 \dots N_\Phi - 1$. Then solution f of the functional equations (14), defined by (24) in the strip $\Pi(-\Phi, \Phi)$, takes the form $f_m(z) := f(z)|_{\vartheta_0 = \vartheta_0^m}$, i.e.*

$$f_m(z) = \frac{f_0^m(z)}{\sin(\mu z)} F(z) S(z)\Big|_{\vartheta_j = \vartheta_j^m,\, j=0,1,2}, \tag{26}$$

has no poles in the closed strip $\Pi[-\Phi + \pi/2, \pi/2 + \Phi)]$ and $f_m \in \mathcal{M}$, where for every m the constants c_0, c_1, c satisfy the linear system (23), (25) of rank two.

4. COMPLETION OF CONSTRUCTION OF THE EIGENVALUES AND EIGENFUNCTIONS

In expression (12) for the solution we should make use of the fact that $f = f_m$ for some m. However, in order to find the corresponding $\varkappa = \varkappa_m$ and then substitute it into the integrand, we have to solve the algebraic equation (19) in Remark 3, $w_0(\varkappa_m) = \sin\vartheta_0^m$, $\vartheta_0^m = \Phi(2m-1)$, $m = 1, 2, \dots N_\Phi - 1$, where w_0 is a root of the polynomial $w^3 - (1+D)w + \nu = 0$, such that $0 < w_0 < 1$. We recall that D and ν in the coefficients of the polynomial depend on $\varkappa$ and on physical parameters, $\nu = \frac{\nu_*}{\varkappa^3}$, $1 + D = \frac{k_M^2 - k^2}{\varkappa^2}$, whereas the right-hand side of the algebraic equation (19) depends on Φ and m via ϑ_0^m.

Assumptions (13) essentially restrict the range of values of the physical parameters k_M, k, ν_* and Φ for which the corresponding real solutions $\varkappa_m$ of the equation $w_0(\varkappa_m) = \sin\vartheta_0^m$ really exist. As we can see, the values $\varkappa_m$ implicitly depend on

$$\epsilon^{-1} := k_M^2 - k^2,$$

ν_* of the polynomial $v^3 - (k_M^2 - k^2)v + \nu_*$, where $v = \varkappa w$, and on $\vartheta_0^m = \Phi(2m-1)$, hence, on the parameters k_M, k, ν_* and Φ. In general case, an efficient study of solutions of the equation $v^3 - (k_M^2 - k^2)v + \nu_* = 0$ can be performed *numerically.*

Let $v_0(\epsilon^{-1}, \nu_*)$, $v_j(\epsilon^{-1}, \nu_*)$, $j = 1, 2$ be the roots of the equation $v^3 - \epsilon^{-1}v + \nu_* = 0$. These roots are obviously connected with the roots w_j, $j = 0, 1, 2$ in (13),

$$w_j(\varkappa) = \frac{v_j}{\varkappa},$$

because $w^3 - \dfrac{\epsilon^{-1}}{\varkappa^2}w + \dfrac{\nu_*}{\varkappa^3} = \displaystyle\prod_{j=0}^{2}\left(w - \frac{v_j}{\varkappa}\right)$.

Introduce the set G_* of such k_M, k, ν_*, Φ that

[6] The rank of the system of linear equations is maximal and equals two.

- it is a subset of the space of 'physical' parameters (k_M, k, ν_*, Φ) satisfying conditions $k_M^2 - k^2 > 0$, $\nu_* > 0$, $0 < \Phi < \pi/2$,
- for all $m = 1, 2, \dots N_\Phi - 1$, where $N_\Phi = \text{ int}\left(\frac{1}{2}\left[\frac{\pi}{2\Phi} + 1\right]\right)$, the following inequalities (see (13)) hold:

$$v_1(\epsilon^{-1}, \nu_*)\,\sin\vartheta_0^m \;>\; v_0(\epsilon^{-1}, \nu_*)\;> 0,$$
$$(-v_2(\epsilon^{-1}, \nu_*))\,\sin\vartheta_0^m \;>\; v_0(\epsilon^{-1}, \nu_*)$$

with $\vartheta_0^m = \Phi(2m-1)$.

Recalling that $0 < w_0(\varkappa_m) = \frac{v_0(\epsilon^{-1},\nu_*)}{\varkappa_m}$ and $w_0(\varkappa_m) = \sin\vartheta_0^m < 1$, we also specify $\varkappa_m$ by the equality

$$\varkappa_m = \frac{v_0(\epsilon^{-1}, \nu_*)}{\sin\vartheta_0^m}\,. \tag{27}$$

In order to show that G_* is a nonempty set, we entertain simple asymptotic analysis of the roots of the polynomial equation

$$v^3 - \epsilon^{-1} v + \nu_* = 0$$

assuming that the parameter $\epsilon := 1/(k_M^2 - k^2)$ is small, i.e. $k_M^2 - k^2 \gg 1$. Traditional asymptotic analysis leads to the approximate expressions of the roots

$$v_0(\epsilon^{-1}, \nu_*) = \nu_*\,\epsilon\,(1 + O(\epsilon)) = \frac{\nu_*}{(k_M^2 - k^2)}(1 + O(1/(k_M^2 - k^2))),$$
$$v_j(\epsilon^{-1}, \nu_*) = (-1)^{j+1}\,\frac{1}{\sqrt{\epsilon}}\,(1 + O(\epsilon)) = (-1)^{j+1}\sqrt{k_M^2 - k^2}\,(1 + O(1/(k_M^2 - k^2))),$$

$j = 1, 2$.[7] We observe that for sufficiently large ϵ^{-1} conditions in the definition of G_* are obviously satisfied and we have

Lemma 4.1. *In the space of 'physical' parameters (k_M, k, ν_*, Φ) the set G_* is not empty.*

We also find an approximate expression for

$$\varkappa_m \;=\; \frac{\nu_*}{(k_M^2 - k^2)\sin\vartheta_0^m}(1 + O(1/(k_M^2 - k^2)))\,.$$

Such simple approximate formulas show that there is a range of parameters k_M, k, ν_* and Φ for which the discrete spectrum is not empty, it consists of $-\varkappa_m^2$, m takes a finite number $N_\Phi - 1$ of values and the corresponding solutions are as follows,

$$u(r, \varphi; \varkappa_m) = \frac{1}{2\pi\ \mathrm{i}}\int\limits_\Gamma \mathrm{d}z\ \ \mathrm{e}^{\varkappa_m r\cos(z)}\,\frac{1}{2}[f_m(z + \varphi) + f_m(z - \varphi)]\,, \tag{28}$$

where in the expression for f_m we took into account that there exist such ϑ_j^m $(j = 0, 1, 2)$ that $w_0(\varkappa_m) = \sin\vartheta_0^m$ and $w_j(\varkappa_m) = \sin\vartheta_j^m$, $\ j = 1, 2$.

It remains to verify that the integral in (28) satisfies the Meixner's condition (5) and is square integrable, (6). The condition (5) is traditionally verified from the estimate for the Sommerfeld transformant $f_m(z) = f_m(\ \mathrm{i}\infty) + O(\exp(\ \mathrm{i}\delta z))$ for some $\delta > 0$, $\ z \to\ \mathrm{i}\infty$. It is worth noticing that the contact condition (5) is also valid because of the estimate $\frac{\partial f_m(z+\varphi)}{\partial\varphi} = O(\exp(\ \mathrm{i}\delta z))$ uniformly with respect to $\varphi \in [0, \Phi]$. One can easily evaluate the Sommerefeld integral as $r \to 0$ in view of the latter estimate.

The asymptotics of $u(r, \varphi; \varkappa_m)$ as $r \to \infty$ is computed by use of the SD method. The exponent in the integrand has two saddle points $\pm\pi$, $(\cos(z))' = 0$. We deform the integration contour into SD paths $\Gamma_{\pm\pi}$ shown in Fig. 3. In the process of such deformation, several poles of the integrand can be intersected and contribute to the asymptotics. We intentionally constructed the Sommerfeld transformant f_m in such a manner that it has no poles in the closed strip $\Pi(-\Phi + \pi/2, \pi/2 + \Phi)$, Fig. 3. The poles z_s^m that can be intersected are symbolically shown in Fig.3 and located in the domain $\pi/2 + \Phi < |\Re(z + \varphi)| < \pi$. We have

$$u(r, \varphi; \varkappa_m) = \sum_s\ \mathrm{res}_{z_s^m = z+\varphi} f_m(z + \varphi)\ \ \mathrm{e}^{\varkappa_m r\cos(z_s^m - \varphi)} + O\left(\frac{\mathrm{e}^{-\varkappa_m r}}{\sqrt{r}}\right), \tag{29}$$

[7] Approximate values of $w_j(\varkappa_m) = \sin\vartheta_j^m$ $(j = 1, 2)$ are $(-1)^{j+1}\frac{\sqrt{k_M^2 - k^2}}{\varkappa_m}$ as $k_M^2 - k^2 \gg 1$.

where the contribution of the saddle point points is

$$\frac{1}{2\pi\ \mathrm{i}} \int\limits_{\Gamma_\pi \cup \Gamma_{-\pi}} \mathrm{d}z\ \ \mathrm{e}^{\varkappa_m r \cos(z)}\, f_m(z+\varphi) \,=\, O\left(\frac{\mathrm{e}^{-\varkappa_m r}}{\sqrt{r}}\right).$$

The asymptotics (28) is nonuniform with respect to φ. In the situation, when a real pole is close to one of the saddle points, the uniform version of the SD method is applied, see e.g. [18]. From the asymptotics (28) we make sure that $u(r,\varphi;\varkappa_m)$ exponentially vanishes and is an eigenfunction corresponding to the eigenvalue $-\varkappa_m^2$. We arrive at the main result,

Theorem 4.1. *There exists a nonempty set G_* of 'physical' values of the parameters k_M, k, ν_* and Φ and such that the problem (1)–(5) has a finite number of classical solutions (eigenfinctions) (28) with the integrand, defined by (26), which satisfy estimate (29) and correspond to the eigenvalues $-\varkappa_m^2$, where $\varkappa_m$ is given by (27), $m = 1, 2\ldots, N_\Phi - 1$ with $N_\Phi =$ int$\left(\frac{1}{2}\left[\frac{\pi}{2\Phi}+1\right]\right)$.*

Remark 5. Now we can assert, taking into account Theorem 4.1, that for some physical parameters, the operator A attributed to the problem has nontrivial discrete component of the spectrum, $E_m = -(k_e^m)^2$, $k_e^m = \sqrt{k^2+\varkappa_m^2}$, $m = 1, 2\ldots N_\Phi - 1$ below the minimum at $E_* = -(k^2 + v_0^2(\epsilon^{-1}, \nu_*))$ of the essential spectrum $\sigma_{ess}(A) = [E_*, \infty)$.

It is worth noticing that in the situation, when an acoustic medium is both inside and outside the wedge bounded by membranes, explicit construction of the eigenfunctions and eigenvalues is hardly available. Nevertheless, we expect that the desired localized solutions exist although the corresponding study is more complex. On the other hand, our approach could be also applied to the problem of existence of localized solutions in the case of angular contact of two thin Kirchhof plates submerged into acoustic medium. The Kirchhof model of a thin plate implies a differential operator of the forth order in the equation for the transverse displacement.

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