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Step wavelets on Vilenkin groups

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Abstract
The construction of wavelet bases and frames on the Vilenkin group G_p is studied. Wavelet systems consisting of functions that are compactly supported and band-limited at the same time are of our interest. A complete description of all refinable functions providing such systems is given.

Keywords Vilenkin groups · Multiresolution analysis · Refinable functions · Step wavelets

1 Introduction

Let G_p denote the locally compact Vilenkin group associated with a positive integer $p \geq 2$ (the case $p = 2$ corresponds to the Cantor group). It is well known that the characters of G_p are the generalized Walsh functions (see [1, 2]). As well as the additive group of the p-adic number field $\mathbb{Q}_p$, the group G_p is a special case of zero-dimensional groups. A general method for constructing the Haar bases on different structures (including local fields of positive characteristic and zero-dimensional groups) was proposed in [3]. It is known [4] that any orthogonal wavelet basis for $L^2(\mathbb{Q}_p)$ consisting of band-limited functions is a "damaged" (wavelet equivalent) Haar basis. The situation is different for the Cantor and Vilenkin groups, where there exist orthogonal band-limited wavelet bases essentially different from the Haar bases. The first examples of such bases were constructed in [5, 6].

Similarly to the real setting, the construction of wavelets on the Vilenkin groups is based on a multiresolution analysis generated by a scaling function that has a number of special properties, in particular, it must be refinable (solution of a refinement equation). Such wavelet systems are called MRA-based (e.g. [7]). MRA-based wavelets inherit important properties (such as smoothness and compactness of the support) of the generating scaling function. That is why we are interested in providing smoothness for refinable functions as much as possible, and the band-limited functions are optimal in this sense. In contrast to the real setting, there exist compactly supported band-limited functions on G_p. The class of such functions is an analogue of the Schwartz class on the real line, and these functions are finite linear combinations of the generalized Walsh functions which, in turn, are step functions. That is why compactly supported band-limited functions are step functions that are used in the theory of approximations on the Vilenkin groups (e.g. [8–12]). A number of concrete examples of compactly supported step scaling functions generating wavelet bases and

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frames on G_p exist in the literature (see [13–16]). Wavelets generated by step scaling functions are also step functions (see "Basic definitions and auxiliary statements" for more details). Also, it is known [17] that step wavelets provide the best order of approximation in some functional spaces.
The goal of the present paper is to give a complete description of all compactly supported step refinable functions on G_p and the corresponding wavelets, in particular, to construct MRA-based tight frames and orthogonal wavelet bases.

2 Basic definitions and auxiliary statements

Given integer $p \geq 2$, the Vilenkin group $G = G_p$ consists of a sequence $x = (x_j)$, where $x_j \in \{0, 1, \dots, p-1\}$, $j \in \mathbb{Z}$, and there exists at most a finite number of negative j such that $x_j \neq 0$. The group operation $\oplus$ on G is defined as the coordinatewise addition modulo p. The topology on G is introduced via the complete system of neighbourhoods of the zero element θ of G

$$U_l := \{(x_j) \in G : \ x_j = 0 \text{ for all } j \leq l\}, \quad U_{l+1} \subset U_l, \quad l \in \mathbb{Z}.$$

As usual, the equality $z = x \ominus y$ means that $z \oplus y = x$. In the case $p = 2$, the group G coincides with the locally compact Cantor group and the subgroup $U := U_0$ is isomorphic to the compact Cantor group, which is the topological Cartesian product of a countable set of cyclic groups with discrete topology. One can show that G is self-dual. The duality pairing on G takes $x,\omega \in G$ to

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_j \omega_{1-j}\right).$$

The Lebesgue spaces on the group G are defined by the Haar measure μ, which is normalized by the condition $\mu(U) = 1$. The Fourier transform of $f \in L^1(G) \cap L^2(G)$ is defined by the formula

$$\hat{f}(\omega) = \int_G f(x)\overline{\chi(x, \omega)}d\mu(x), \quad \omega \in G,$$

and extends to the space $L^2(G)$ in the standard way. As in previous works (e.g. [14, 18, 19]), we define an automorphism $A : G \to G$ by letting $(Ax)_j = x_{j+1}$ for all $x = (x_j) \in G$. The following properties are well known (see, e.g., [1, 2, 20]):

$$\int_G f(x \ominus y)\overline{\chi(x, \omega)}d\mu(x) = \overline{\chi(y, \omega)}\hat{f}(\omega), \tag{1}$$

$$\int_G f(A^l x)\overline{\chi(x, \omega)}d\mu(x) = p^{-l}\hat{f}(A^{-l}\omega), \tag{2}$$

$$\int_G \mathbf{1}_U(A^l\omega)\chi(x, \omega)d\mu(\omega) = p^{-l}\mathbf{1}_U(A^{-l}x), \tag{3}$$

where $l \in \mathbb{Z}$, $y \in G$ and $\mathbf{1}_U$ denotes the characteristic function of U.
Let $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{Z}_+ := \{0, 1, \dots\}$. The mapping $\lambda : G \to \mathbb{R}_+$ is defined by

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \qquad x = (x_j) \in G.$$

The image of the discrete subgroup

$$H := \{(x_j) \in G : \ x_j = 0 \text{ for all } j > 0\}$$

under λ is the set of nonnegative integers: $\lambda(H) = \mathbb{Z}_+$. For each $k \in \mathbb{Z}_+$, let $h_{[k]}$ denote the element of H such that $\lambda(h_{[k]}) = k$. The *Walsh functions* W_k, $k \in \mathbb{Z}_+$, on U are defined by

$$W_k(x) = \chi(x, h_{[k]}), \quad x \in U.$$

The functions W_k can be extended to the group G by the H-periodicity property as follows:

$$W_k(x \oplus h) = W_k(x) \quad \text{for all} \quad h \in H.$$

It is well known that $\{W_k\}_{k=0}^{\infty}$ is an orthonormal basis for $L^2(U)$.
For any integer m, we set

$$U_{m,s} := A^{-m}(h_{[s]}) \oplus A^{-m}(U), \quad s \in \mathbb{Z}_+.$$

It is easy to check that, for every $0 \le k \le p^m - 1$, the function W_k is constant on each $U_{m,s}$, with $0 \le s \le p^m - 1$. Note also that $A^{-m}(U) = U_m$, $\lambda(U_m) = [0,p^{-m}]$, $\lambda(U_{m,s}) = [sp^{-m},(s+1)p^{-m}]$, and $\mu(U_m) = \mu(U_{m,s}) = p^{-m}$. It follows from the definitions that for any $x \in A^m(U)$, we have

$$W_k(A^{-m}x) = \chi(A^{-m}x, h_{[k]}) = \chi(x, A^{-m}h_{[k]}), \quad k \in \mathbb{Z}_+. \tag{4}$$

We write $f \in \mathcal{S}^{(m)}$, if f is constant on $U_{m,s}$ for each $s \in \mathbb{Z}_+$. We will say that f is a *step function*, if $f \in \mathcal{S}^{(m)}$ for some m. Further, let

$$\mathcal{S}_l^{(m)} := \{f \in \mathcal{S}^{(m)} : \ \operatorname{supp} f \subset A^l(U)\}.$$

The set $\mathcal{S}_l^{(m)}$ is an analogue of the Schwartz class on the real line (cf [12, 21]).

Proposition 1 The following properties hold:

(a) if $f \in \mathcal{S}^{(m)} \cap L^1(G)$, then $\operatorname{supp} \hat{f} \subset A^m(U)$;
(b) if $f \in L^1(G)$ and $\operatorname{supp} f \subset A^m(U)$, then $\hat{f} \in \mathcal{S}^{(m)}$.

Proposition 2 For any $m, l \in \mathbb{Z}$, we have

$$f \in \mathcal{S}_l^{(m)} \iff \hat{f} \in \mathcal{S}_m^{(l)}. \tag{5}$$

For the proofs of these two propositions, see [2, Sect. 6.2] and [22].
A function $\varphi \in L^2(G)$ is called *refinable* if it satisfies the following *refinement equation*:

$$\varphi(x) = p \sum_{k=0}^{\infty} a_k \varphi(Ax \ominus h_{[k]}), \tag{6}$$

where $a_k \in \mathbb{C}$.
It is proved in [18] that the sum in (6) is finite whenever φ is compactly supported (in contrast to the real setting). In the present paper, we are interested in compactly supported refinable functions. Thus, we will consider refinement equations of the following type:

$$\varphi(x) = p \sum_{k=0}^{p^n-1} a_k \varphi(Ax \ominus h_{[k]}), \quad x \in G. \tag{7}$$

Applying (1) and (2), we can rewrite (7) in the Fourier domain as follows:

$$\hat{\varphi}(\omega) = m_0(A^{-1}\omega)\hat{\varphi}(A^{-1}\omega), \tag{8}$$

where

$$m_0(\omega) = \sum_{k=0}^{p^n-1} a_k \overline{W_k(\omega)}, \quad \omega \in G. \tag{9}$$

The function m_0 is said to be the *mask* of φ.

Proposition 3 A function m_0 is given by (9) if and only if it is H-periodic and belongs to $\mathcal{S}^{(n)}$.

This proposition is easily deduced from the well known properties of the Walsh functions (see, e.g. [2, Sect. 1.3]). Further, the next statement immediately follows from (8) and [21, Theorem 1].

Proposition 4 Suppose that $m_0 \in \mathcal{S}^{(n)}$ is the mask of a compactly supported refinable function φ. If $\widehat{\varphi}(\theta) \neq 0$, then

$$\widehat{\varphi}(\omega) = \widehat{\varphi}(\theta) \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \tag{10}$$

where the product is finite for each $\omega \in G$.

Corollary 1 If φ is a refinable step function satisfying (7) and $\widehat{\varphi}(\theta) = 1$, then

$$\widehat{\varphi}(\omega) = \mathbf{1}_{U_{n-1}}(\omega) + \sum_{l} \widehat{\varphi}(A^{1-n}h_{[l]})\mathbf{1}_{U_{n-1}\oplus A^{1-n}h_{[l]}}(\omega), \tag{11}$$

where the sum is finite.

Proof It follows from (9) and Proposition 3 that $m_0 \in S^{(n)}$. Due to (10), this obviously yields $\widehat{\varphi} \in S^{(n-1)}$. Since $\varphi \in S_m$ for some m, by Proposition 1, we have $\widehat{\varphi} \in S_m^{(n-1)}$ which implies (11).

Proposition 5 Suppose that the Fourier transform of a function $\varphi \in L^2(G)$ can be written as

$$\widehat{\varphi}(\omega) = \mathbf{1}_{U_{n-1}}(\omega) + \sum_{l=1}^{L} d_l \mathbf{1}_{U_{n-1}\,\oplus\, A^{1-n}h_{[l]}}(\omega), \quad \omega \in G, \tag{12}$$

where L is a positive integer and $d_l \in \mathbb{C}$ for $l \in \{1, \dots, L\}$. Then

$$\varphi(x) = (1/p^{n-1})\mathbf{1}_U(A^{1-n}x)(1 + \sum_{l=1}^{L} d_l W_l(A^{1-n}x)), \quad x \in G. \tag{13}$$

Proof First note that, using (3), we get

$$\int_G \mathbf{1}_{U_{n-1}}(\omega \ominus A^{1-n}h_{[l]})\chi(x,\omega)d\mu(\omega) = \int_G \mathbf{1}_{U_{n-1}}(\omega)\chi(x,\omega \oplus A^{1-n}h_{[l]})d\mu(\omega)$$

$$= \chi(x, A^{1-n}h_{[l]})\int_{U_{n-1}} \chi(x,\omega)d\mu(\omega) = (1/p^{n-1})\mathbf{1}_U(A^{1-n}x)W_l(A^{1-n}x). \tag{14}$$

Thus, taking the inverse Fourier transform of (12), we obtain (13).

The refinable functions and their masks are needed for the construction of MRA-based wavelets, in particular, wavelet bases and frames.
Given $\Psi = \{\psi^{(1)}, \dots, \psi^{(r)}\} \subset L^2(G)$, using the notation

$$f_{j,k}(x) := p^{j/2} f(A^j x \ominus h_{[k]}), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}_+,$$

we say that

$$X(\Psi) := \{\psi_{j,k}^{(\nu)} : \ 1 \le \nu \le r, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}_+\}.$$

is a *wavelet system* generated by the *wavelet functions* $\psi^{(\nu)}$. A wavelet system $X(\Psi)$ is a Parseval frame (or a *wavelet normalized tight frame*) for $L^2(G)$, if

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} \sum_{\nu=1}^{r} |\langle f, \psi_{j,k}^{(\nu)}\rangle|^2 = \|f\|^2$$

for all $f \in L^2(G)$. This is equivalent to $f = \sum_{g\in X(\Psi)} \langle f, g\rangle g$ for all $f \in L^2(G)$.
The following algorithmic method (see [14, Sect. 7.4], [18]) allows us to construct a tight wavelet frame from any suitable mask m_0.
Choose a Walsh polynomial

$$m_0(\omega) = \sum_{k=0}^{p^n-1} a_k \overline{W_k(\omega)}, \tag{15}$$

such that

$$m_0(\theta) = 1, \quad \sum_{l=0}^{p-1} \left| m_0\big(\omega \oplus \delta_l\big)\right|^2 \le 1, \quad \omega \in G, \tag{16}$$

where $\delta_l \in U$, $\lambda(\delta_l) = l/p$, $l = 0, \dots, p-1$. Define $\varphi \in L^2(G)$ by

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G.$$

Given $r \ge p-1$, one can find Walsh polynomials

$$m_\nu(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^{(\nu)} \overline{W_\alpha(\omega)}, \quad 1 \le \nu \le r,$$

such that the rows of the matrix

$$M(\omega) := \begin{bmatrix} m_0(\omega) & m_1(\omega) & \dots & m_r(\omega) \\ m_0(\omega\oplus\delta_1) & m_1(\omega\oplus\delta_1) & \dots & m_r(\omega\oplus\delta_1) \\ \dots & \dots & \dots & \dots \\ m_0(\omega\oplus\delta_{p-1}) & m_1(\omega\oplus\delta_{p-1}) & \dots & m_r(\omega\oplus\delta_{p-1}) \end{bmatrix}$$

form an orthonormal system for each $\omega \in G$. The explicit formulas for the implementation of this step are given in [18].
Setting

$$\psi^{(\nu)}(\omega) = p \sum_{\alpha=0}^{p^n-1} a_\alpha^{(\nu)} \varphi(A\omega \ominus h_{[\alpha]}), \quad 1 \le \nu \le r,$$

we obtain a set of wavelet functions $\psi^{(1)}, \dots, \psi^{(r)}$ generating a Parseval frame. Obviously, if φ is a compactly supported step function, then the functions $\psi^{(\nu)}$, as well as their shifts and scales $\psi_{j,k}^{(\nu)}$, inherit the same properties.
Thus, to construct a wavelet tight frame consisting of compactly supported step functions, we need a Walsh polynomial m_0 that is the mask of a refinable step function and such that (16) is satisfied. It will be clear from "Refinable step functions" that a suitable set of zeros and small enough (in modulus) non-zero values of m_0 guarantee these properties. Note that if all values of m_0 are known, then the coefficients a_k in (15) can be found using the Vilenkin-Chrestenson transform (see, e.g., [15, 18]).
A refinable function φ is called *orthogonal* if its H-shifts $\varphi(\cdot \oplus h)$, $h \in H$, form an orthonormal system. If wavelet functions

$\psi^{(1)}, \dots, \psi^{(p-1)}$ generate a Parseval frame constructed from the mask of an orthogonal refinable function, then such a wavelet system is an orthonormal basis for $L^2(G)$ (see, e.g., [14, 15]).
The following methods are used to check the orthogonality of the function φ: (a) the modified Cohen criterion (see [14, Theorem 5.2], [17, Sect. 2.4], and [18, Theorem 8.2.2]), (b) the blocking sets criterion (see [6, 22]), and (c) the N-valid trees method (see [25]). In view of (5) and (11), it will be more convenient for us to use the following criterion.

Proposition 6 A refinable function φ is orthogonal if and only if

$$\sum_{h\in H} |\hat{\varphi}(\omega \oplus h)|^2 = 1 \qquad \text{for a. e.} \quad \omega \in G.$$

The proof of this criterion is given in [19, Sect. 3]; a similar criterion for the integer shifts of functions in $L^2(\mathbb{R})$ is well known (see, e.g. [7, Proposition 1.1.12]).

Example 1 Let $p=2$, $n=3$. Then from (7) we have

$$\varphi(x) = 2\sum_{k=0}^{7} a_k \varphi(Ax \ominus h_{[k]}), \quad x \in G. \tag{17}$$

Suppose that the coefficients $a_0, a_1, \dots, a_7$ are defined as follows:

$$a_0 = \frac{1}{8}(1+a+b+c+\alpha+\beta+\gamma), \quad a_1 = \frac{1}{8}(1+a+b+c-\alpha-\beta-\gamma),$$

$$a_2 = \frac{1}{8}(1+a-b-c+\alpha-\beta-\gamma), \quad a_3 = \frac{1}{8}(1+a-b-c-\alpha+\beta+\gamma),$$

$$a_4 = \frac{1}{8}(1-a+b-c-\alpha+\beta-\gamma), \quad a_5 = \frac{1}{8}(1-a+b-c+\alpha-\beta+\gamma),$$

$$a_6 = \frac{1}{8}(1-a-b+c-\alpha-\beta+\gamma), \quad a_7 = \frac{1}{8}(1-a-b+c+\alpha+\beta-\gamma),$$

where

$$|a|^2 + |\alpha|^2 = |b|^2 + |\beta|^2 = |c|^2 + |\gamma|^2 = 1.$$

If $a=0$ or $c=0$, then by [15, Theorema 2.2] the refinable function φ provides a Parseval frame for $L^2(G)$. If a and c differ from zero, then φ generates an orthogonal MRA-wavelets in $L^2(G)$. In particular, for $a=c=1$, $0 \le |b| < 1$, we have the orthogonal refinable step function

$$\varphi(x) = (1/4)\mathbf{1}_U(y)(1 + W_1(y) + bW_2(y) + W_3(y) + \beta W_6(y)), \tag{18}$$

where $y = A^{-2}x$; see [19, Example 1.5], [5, Example 4], [14, Example 5.3], [6, Example 3], and [23, Example 2.32].
Other examples of refinable step functions providing orthogonal wavelet bases on the Vilenkin groups are given in [25–28].

3 Refinable step functions

As is explained in "Basic definitions and auxiliary statements", to construct a tight wavelet frame consisting of compactly supported step functions it suffices to have a refinable function which possesses the same properties and such that its mask m_0 satisfies (16). Recall also that such a frame is an orthogonal basis whenever the refinable function is orthogonal. The construction of the required refinable function will be discussed in this section.

Now, it will be more convenient for us to use the following notation. If $x = (x_j) \in G$, $x_j = 0$ for all $j < -m$, $m \in \mathbb{Z}_+$, we will write $x = x_{-m}x_{-m+1}\dots x_0, x_1x_2 \dots$. In particular, each $x \in U$ can be written as $x = 0, x_1x_2 \dots$. If $x_j = 0$ whenever $j > n$, then we write $x = x_{-m}x_{-m+1}\dots x_0, x_1x_2 \dots x_n$. So, $x = x_{-m}x_{-m+1}\dots x_0$ if $x_j = 0$ for all $j > 0$, i.e. $x \in H$. Also, we note that $A(0, x_1x_2 \dots x_n) = x_1, x_2 \dots x_n$ and $A^{-1}(0, x_1 \dots x_n) = 0, 0x_1, \dots x_n$.
Let us denote by $\mathscr{M}_0^{(n)}$ the set of H-periodic functions m_0 such that $m_0 \in \mathcal{S}^{(n)}$ and $m_0(\theta) = 1$. Suppose that $m_0 \in \mathscr{M}_0^{(n)}$. Then

$$m_0(\omega) = m_0(0, \omega_1 \dots \omega_n) \tag{19}$$

for each $\omega \in G$. Setting

$$b[s_1 \dots s_n] := m_0(0, s_1 \dots s_n),$$

where $s_j \in \{0, \dots, p-1\}$, we see that the complete set of values $b[s_1 \dots s_n]$ defines such a function m_0. Further, if m_0 is the mask of a compactly supported refinable function φ, then, by Proposition 4, for each $\omega = \omega_{-m}\omega_{-m+1}\dots\omega_0, \omega_1\omega_2 \dots$, we have

$$\hat{\varphi}(\omega) = b[\omega_0 \dots \omega_{n-1}]b[\omega_{-1} \dots \omega_{n-2}] \dots b[0 \dots 0\omega_{-m}]. \tag{20}$$

Definition 1 Let $m_0 \in \mathscr{M}_0^{(n)}$. Given $r \in \mathbb{Z}_+$, we denote by $\sigma_r = \sigma_r(m_0)$ the set of the vectors

$$(s_0, s_1, \dots, s_r), \quad s_0 \neq 0, \quad s_j \in \{0, 1, \dots, p-1\},$$

such that for every $l \in \{0, 1, \dots, r\}$, there holds

$$b[s_{1-n+l} \dots s_l] \neq 0, \tag{21}$$

where $s_j = 0$ whenever $j < 0$. We denote by $\sigma_\infty = \sigma_\infty(m_0)$ the set of the sequences

$$(s_0, s_1, \dots), \quad s_0 \neq 0, \quad s_j \in \{0, 1, \dots, p-1\},$$

such that (21) holds for any $l \geq 0$. Also, we denote by $\sigma^* = \sigma^*(m_0)$ the set of the vectors

$$(s_1, \dots, s_M), \quad M \geq n, \quad s_j \in \{0, 1, \dots, p-1\},$$

such that (21) holds for any $l \in \{n, n+1, \dots, M\}$.

It is easily seen that if $\sigma_r = \varnothing$, then $\sigma_{r+1} = \varnothing$, and if $\sigma_\infty \neq \varnothing$, then $\sigma_r \neq \varnothing$ for every r.

Theorem 1 Let $r \in \mathbb{Z}_+$ and $m_0 \in \mathcal{M}_0^{(n)}$, where $n \in \mathbb{N}$. For m_0 to be the mask of a refinable function $\varphi \in S_{n-1}^{(r-n+1)}$, it is necessary and sufficient that $\sigma_r = \varnothing$.

Proof The function $g(\omega) := \prod_{k=1}^{\infty} m_0(A^{-k}\omega)$ is finite for every ω because

$$m_0(A^{-k}\omega) = m_0(\theta) = 1$$

whenever k is large enough. Obviously, g belongs to the space $\mathcal{S}^{(n-1)}$, since $m_0 \in \mathcal{S}^{(n)}$. With these in hand, it is easy to see that Theorem 1 follows from the following lemma.

Lemma 1 Let r, n and m_0 be as in Theorem 1. Suppose that

$$g(\omega) = \prod_{k=1}^{\infty} m_0(A^{-k}\omega), \quad \omega \in G.$$

Then supp $g \subset A^{r-n+1}(U)$ if and only if $\sigma_r = \varnothing$.

Proof First of all, recall that $g(\omega)$ is finite for every ω. Suppose that $\sigma_r = \varnothing$ and there exists $\omega \notin A^{r-n+1}(U)$ such that $g(\omega)\neq 0$. If now $\omega = A^{r-n+l+1}(s_0, s_1 s_2 \dots)$, where $l \in \mathbb{Z}_+$, $s_0 \neq 0$, $s_j \in \{0, 1, \dots, p-1\}$, then $m_0(A^{-k}\omega) \neq 0$ for all $k \in \mathbb{N}$. Due to (19), for every positive integer $k \le r+1$, we have

$$m_0(A^{-k-l}\omega) = m_0\left(A^{r+1-n-k}(s_0, s_1 s_2 \dots)\right) = b[s_{r-n-k+2}\dots s_{r-k+1}] \neq 0,$$

where $s_j = 0$ whenever $j < 0$. Hence, the vector $(s_0, s_1, \dots, s_r)$ belongs to σ_r, which contradicts our assumption $\sigma_r = \varnothing$. Thus, we obtain $\operatorname{supp} g \subset A^{r-n+1}(U)$.

Conversely, let $\operatorname{supp} g \subset A^{r-n+1}(U)$. Suppose that $\sigma_r \neq \varnothing$, i.e. there exists

$$(s_0, s_1, \dots, s_r) \in \sigma_r.$$

Set $\omega = A^{r-n+1}(s_0, s_1 \dots s_r)$. Since $\omega \notin \operatorname{supp} g$, we have $g(\omega) = 0$. Hence, there exists $k \in \mathbb{N}$ such that $m_0(A^{-k}\omega) = 0$. However, if $1 \le k \le r+1$, then, using (19), we obtain

$$m_0(A^{-k}\omega) = m_0(A^{r-n-k+1}(s_0, s_1 \dots s_2)) = b[s_{r-k-n+2} \dots s_{r-k+1}] \neq 0$$

and $m_0(A^{-k}\omega) = m_0(\theta) = 1$ if $k > r+1$, which contradicts our assumption. Thus $\sigma_r = \varnothing$.

Corollary 2 Let n and m_0 be as in Theorem 1. Suppose that $r \in \{1, \dots, n\}$ and $m_0(\omega) = 0$ for every $\omega \in A^{-r+1}(U) \setminus A^{-r}(U)$. Then m_0 is the mask of a refinable function $\varphi \in S_{n-1}^{(1-r)}$. Moreover, if $m_0(\omega) = 1$ for every $\omega \in A^{-r}(U)$, then $\varphi = p^{1-r}\mathbf{1}_{A^{r-1}(U)}$.

Proof Due to Theorem 1, to prove the first statement, it suffices to verify that $\sigma_{n-r} = \varnothing$. Suppose that $(s_0, \dots s_{n-r}) \in \sigma_{n-r}$, then

$$m_0(A^{-r}(s_0, s_1 \dots s_{n-r})) = m_0(0, 0 \dots 0 s_0 \dots s_{n-r}) \neq 0,$$

but this is not true because $s_0 \neq 0$, and hence $A^{-r}(s_0, s_1 \dots s_{n-r}) \in A^{-r+1}(U) \setminus A^{-r}(U)$.

Let now $m_0(\omega) = 1$ for every $\omega \in A^{-r}(U)$. If $\omega \in \operatorname{supp} \widehat{\varphi} = A^{1-r}(U)$, i.e. $\omega = A^{1-r}(0, s_0 \dots s_n)$, then $A^{-\nu}\omega \in A^{-r}(U)$ for every $\nu \in \mathbb{N}$, which yields $m_0(A^{-\nu}\omega) = 1$. It follows that $\widehat{\varphi} = \mathbf{1}_U(A^{r-1}\cdot)$, which yields

$$\varphi(x) = p^{1-r}\mathbf{1}_U(A^{1-r}x) = p^{1-r}\mathbf{1}_{A^{r-1}(U)}(x).$$

Lemma 2 Suppose that $m_0 \in \mathcal{M}_0^{(n)}$ is the mask of a compactly supported refinable step function φ. Then $\sigma_\infty = \emptyset$.

Proof Assume that σ_∞ contains a sequence $(s_0, s_1, \dots)$ with $s_0 \neq 0$. Then, by (20),

$$\begin{aligned} &\widehat{\varphi}(s_0 \dots s_M, s_{M+1} \dots s_{M+n-1}) \\ &= b[s_M \dots s_{M+n-1}]b[s_{M-1} \dots s_{M+n-2}] \dots b[0 \dots 0 s_0] \neq 0 \end{aligned}$$

for every $M \in \mathbb{N}$. Thus, the function $\widehat{\varphi}$ is not compactly supported, which, due to Proposition 2, contradicts our assumption that φ is a step function.

The following theorem refines Theorem 3.9 in [27].

Theorem 2 Let $n \in \mathbb{N}$, $m_0 \in \mathcal{M}_0^{(n)}$, and $N = p^{n-1} - 1$. For m_0 to be the mask of a compactly supported refinable step function, it is necessary and sufficient that $\sigma_N = \varnothing$.

Proof The sufficiency follows from Theorem 1. To prove the necessity, we assume that m_0 is the mask of a compactly supported refinable step function φ. Suppose that $\sigma_N \neq \varnothing$, i.e. there exists a vector $(s_0, \dots, s_N)$ belonging to σ_N. If $s_{l+1} = 0, \dots, s_{l+n-1} = 0$ for some $l \le N - n + 1$, then

$$(s_0, \dots, s_l, 0, 0, \dots) \in \sigma_\infty,$$

that is not true by Lemma 2. Since there are at most N vectors $(s_{k+1},\cdots s_{k+n-1})$, $s_j \in \{0,1,\dots,p-1\}$, different from $(0,\dots,0)$, there exist k_1 and k_2 such that $-n+1<k_1<k_2\le N-n+2$ and

$$s_{k_1+j}=s_{k_2+j},\quad j=0,\dots n-2. \tag{22}$$

If $k_2-k_1>n-1$, then the vector $(s_{k_1},\dots,s_{k_2-1},s_{k_2},\dots,s_{k_2+n-2})$ is a part of

$$(s_{-n+2},\dots,s_{-1},s_0,\dots,s_N),$$

and it belongs to σ^*. Hence, taking into account (22), we have

$$(s_0,\dots,s_{k_1},\dots s_{k_2-1},s_{k_1},\dots,s_{k_2-1},s_{k_1},\dots,s_{k_2-1},s_{k_1},\dots,s_{k_2-1},\dots)\in\sigma_\infty \tag{23}$$

whenever $k_1\ge 0$, and

$$(s_0,\dots,s_{k_2-1},s_{k_1},\dots,s_{k_2-1},s_{k_1},\dots,s_{k_2-1},s_{k_1},\dots,s_{k_2-1},\dots)\in\sigma_\infty. \tag{24}$$

whenever $k_1<0$. Due to Lemma 2, this contradicts our assumptions.

Let now $k_2-k_1\le n-1$, and let L be the maximal integer such that $L(k_2-k_1)\le n-1$. Since the vector

$$(s_{k_1},\dots,s_{k_2-1},s_{k_1},\dots,s_{k_2-1},\dots,s_{k_1},\dots,s_{k_2-1}),$$

where $s_{k_1},\dots,s_{k_2-1}$ is taken $L+1$ times, is a part of

$$(s_{-n+2},\dots,s_{-1},s_0,\dots,s_N),$$

and it belongs to σ^*. Hence, again (23) and (24) hold, which, due to Lemma 2, contradicts our assumptions.

Corollary 3 Suppose that m_0 and N are as in Theorem 2. For m_0 to be the mask of a compactly supported refinable step function, it is necessary and sufficient that

either $b[0\cdots 0s_0]=0$ for every $s_0\in\{1,\dots,p-1\}$;

or $b[0\cdots 0s_0s_1]=0$ for every $s_1\in\{0,\dots,p-1\}$ instead of $b[0\cdots 0s_0]=0$ for some s_0;

or $b[0\cdots 0s_0s_1s_2]=0$ for every $s_2\in\{0,\dots,p-1\}$ instead of $b[0\cdots 0s_0s_1]=0$ for some vector (s_0,s_1);

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or $b[s_{l-n+2}\dots s_{l+1}]=0$ for every $s_{l+1}\in\{0,\dots,p-1\}$ instead of $b[s_{l-n+1}\dots s_l]=0$ for some vector $(s_{l-n+1},\dots,s_l)$, $l<N$.

Corollary 3 provides a complete description of all masks for compactly supported refinable step functions. Let us illustrate this by examples. We will use Theorems 1 and 2, Lemma 2 and formula (20) in our arguments.

Example 2 Let $p=3$ and $n=2$.

(a) If $b[01]=b[02]=0$, then (1),(2) $\notin\sigma_0$ and so $\sigma_0=\varnothing$. Hence, $\hat\varphi\in S_{-1}^{(1)}$ and $\hat\varphi=\mathbf{1}_{A^{-1}(U)}$. By (3), it follows that

$$\varphi(x)=(1/3)\mathbf{1}_U(A^{-1}x),\quad x\in G.$$

(b) If $b[01]=b[20]=b[21]=b[22]=0$, then $\sigma_1=\varnothing$ (because $(1,s)$, $(2,s)\notin\sigma_1$ for any $s\in\{0,1,2\}$). Hence, $\hat\varphi\in S_0^{(1)}$, which yields

$$\hat\varphi=\mathbf{1}_{A^{-1}(U)}+\hat\varphi(0,1)\mathbf{1}_{A^{-1}(U)\oplus 0,1}+\hat\varphi(0,2)\mathbf{1}_{A^{-1}(U)\oplus 0,2}$$

$$=\mathbf{1}_{A^{-1}(U)}+b[01]\mathbf{1}_{A^{-1}(U)\oplus 0,1}+b[02]\mathbf{1}_{A^{-1}(U)\oplus 0,2},$$

where $b[01]=0$. By Proposition 5, it follows that

$$\varphi(x)=(1/3)\mathbf{1}_U(A^{-1}x)(1+b[02]W_2(A^{-1}x)).$$

(c) If $b[02] = b[10] = b[11] = b[12] = 0$, then similarly to (b) (changing the roles of 1 and 2), we have

$$\varphi(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + b[01]W_1(A^{-1}x)).$$

(d) If $b[20] = b[22] = b[10] = b[11] = b[12] = 0$, then $\sigma_2 = \varnothing$. Indeed, obviously, $(1, s, s'), (2, s, s') \notin \sigma_2$ for all $s, s' \in \{0, 1, 2\}$. Hence, $\hat{\varphi} \in S_1^{(1)}$, which yields

$$\begin{aligned}\hat{\varphi} &= \mathbf{1}_{A^{-1}(U)} + \hat{\varphi}(0,1)\mathbf{1}_{A^{-1}(U)\oplus 0,1} + \hat{\varphi}(0,2)\mathbf{1}_{A^{-1}(U)\oplus 0,2}\\ &+\hat{\varphi}(1)\mathbf{1}_{A^{-1}(U)\oplus 1} + \hat{\varphi}(1,1)\mathbf{1}_{A^{-1}(U)\oplus 1,1} + \hat{\varphi}(1,2)\mathbf{1}_{A^{-1}(U)\oplus 1,2}\\ &+\hat{\varphi}(2)\mathbf{1}_{A^{-1}(U)\oplus 2} + \hat{\varphi}(2,1)\mathbf{1}_{A^{-1}(U)\oplus 2,1} + \hat{\varphi}(2,2)\mathbf{1}_{A^{-1}(U)\oplus 2,2}\\ &= \mathbf{1}_{A^{-1}(U)} + b[01]\mathbf{1}_{A^{-1}(U)\oplus 0,1} + b[02]\mathbf{1}_{A^{-1}(U)\oplus 0,2}\\ &+b[01]b[10]\mathbf{1}_{A^{-1}(U)\oplus 1} + b[01]b[11]\mathbf{1}_{A^{-1}(U)\oplus 1,1} + b[01]b[12]\mathbf{1}_{A^{-1}(U)\oplus 1,2}\\ &+b[02]b[20]\mathbf{1}_{A^{-1}(U)\oplus 2} + b[02]b[21]\mathbf{1}_{A^{-1}(U)\oplus 2,1} + b[02]b[22]\mathbf{1}_{A^{-1}(U)\oplus 2,2}.\end{aligned}$$

By Proposition 5, it follows that

$$\begin{aligned}\varphi(x) &= (1/3)\mathbf{1}_U(A^{-1}x)(1 + b[01]W_1(A^{-1}x)\\ &+b[02]W_2(A^{-1}x) + b[02]b[21]W_7(A^{-1}x)).\end{aligned} \tag{25}$$

Applying Proposition 6, it is easy to verify that the scaling function φ is orthogonal if $|b[01]|^2 + |b[21]|^2 = 1$ and $|b[02]| = 1$.

(e) If $b[10] = b[11] = b[20] = b[21] = b[22] = 0$, then similarly to (d) (changing the roles of 1 and 2), we have

$$\begin{aligned}\varphi(x) &= (1/3)\mathbf{1}_U(A^{-1}x)\Big(1 + b[01]W_1(A^{-1}x)\\ &+b[02]W_2(A^{-1}x) + b[01]b[12]W_5(A^{-1}x)\Big).\end{aligned} \tag{26}$$

This scaling function is orthogonal if $|b[01]| = 1$ and $|b[02]|^2 + |b[12]|^2 = 1$.

Let us show that all refinable step functions for $p = 3$, $n = 2$ are described in items (a) - (e) of Example 2. The case $b[01] = b[02] = 0$ is considered in (a). Suppose that $b\,[02] \neq 0$, $b[01] = 0$. Then $b[20] = 0$ (otherwise $(2, 0, 0, ...) \in \sigma_\infty$), $b[22] = 0$ (otherwise $(2, 2, 2, ...) \in \sigma_\infty$) and either $b[12] = 0$ (otherwise $(1, 2, 1, 2, ...) \in \sigma_\infty$) or $b[21] = 0$ (otherwise $(2, 1, 2, 1, ...) \in \sigma_\infty$). If $b[21] = 0$, we are under assumptions of item (b). If $b[12] = 0$ and $b[21] \neq 0$, then $b[10] = b[11] = 0$ (because otherwise $(2, 1, 0, 0, 0, ...) \in \sigma_\infty$ or $(2, 1, 1, 1, ...) \in \sigma_\infty$), and hence the assumptions of (d) are satisfied. Similarly, if $b[02] = 0$, $b[01] \neq 0$, then we are under assumptions of (c) or (e). If $b[02] \neq 0$, $b[01] \neq 0$, then repeating the above arguments, we obtain the assumptions of (d) or (e).

In what follows we will provide $\hat{\varphi}$ explicitly for each step function φ. The explicit form for φ itself may be extracted immediately from this due to Proposition 5.

Example 3 Let $p = 2$ and $n = 3$.

(a) If $b[001] = 0$, then $(1) \notin \sigma_0$, and so $\sigma_0 = \varnothing$. Hence, $\hat{\varphi} \in S_{-2}^{(2)}$, which yields

$$\hat{\varphi} = \mathbf{1}_{A^{-2}(U)}.$$

(b) If $b[010] = b[011] = 0$, then $\sigma_1 = \varnothing$ (because $(1,1) \notin \sigma_1$ and $(1,0) \notin \sigma_1$). Hence, $\hat{\varphi} \in S_{-1}^{(2)}$, which yields

$$\hat{\varphi} = \mathbf{1}_{A^{-2}(U)} + \hat{\varphi}(0,01)\mathbf{1}_{A^{-2}(U)\oplus 0,01} = \mathbf{1}_{A^{-2}(U)} + b[001]\mathbf{1}_{A^{-2}(U)\oplus 0.01}.$$

(c) If $b[010] = b[110] = b[111] = 0$, then $\sigma_2 = \varnothing$ (because $(1,1,0) \notin \sigma_2$, $(1,1,1) \notin \sigma_2$ and $(1,0,s) \notin \sigma_2$, $s \in \{0,1\}$). Hence, $\hat{\varphi} \in S_0^{(2)}$, which yields

$$\hat{\varphi} = \mathbf{1}_{A^{-2}(U)} + \hat{\varphi}(0,01)\mathbf{1}_{A^{-2}(U)\oplus 0,01} + \hat{\varphi}(0,1)\mathbf{1}_{A^{-2}(U)\oplus 0,1} + \hat{\varphi}(0,11)\mathbf{1}_{A^{-2}(U)\oplus 0,11}.$$

Since $\hat{\varphi}(0,01) = b[001]$, $\hat{\varphi}(0,1) = b[001]b[010] = 0$, $\hat{\varphi}(0,11) = b[001]b[011]$, we conclude

$$\widehat{\varphi}(\omega) = \mathbf{1}_{A^{-2}(U)}(\omega) + b[001]\mathbf{1}_{A^{-2}(U)\,\oplus\,0{,}01} + b[001]b[011]\mathbf{1}_{A^{-2}(U)\,\oplus\,0{,}011}.$$

(d) If $b[011] = b[100] = b[101] = 0$, then $\sigma_2 = \varnothing$ (because $(1{,}0{,}0) \notin \sigma_2$, $(1{,}0{,}1) \notin \sigma_2$ and $(1{,}1{,}s) \notin \sigma_2$, $s \in \{0{,}1\}$). Hence $\widehat{\varphi} \in S_0^{(2)}$, which yields

$$\widehat{\varphi} = \mathbf{1}_{A^{-2}(U)} + \widehat{\varphi}(0,01)\mathbf{1}_{A^{-2}(U)\oplus 0.01} + \widehat{\varphi}(0,1)\mathbf{1}_{A^{-2}(U)\oplus 0.1} + \widehat{\varphi}(0,10)\mathbf{1}_{A^{-2}(U)\oplus 0.1},$$

and since $\widehat{\varphi}(0,01) = b[001]$, $\widehat{\varphi}(0,1) = b[001]b[010] = 0$, $\widehat{\varphi}(0,11) = b[001]b[011] = 0$, we have

$$\widehat{\varphi} = \mathbf{1}_{A^{-2}(U)} + b[001]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.01} + b[001]b[010]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.1}.$$

(e) Let $b[100] = b[101] = b[111] = 0$. Note that this assumption is necessary to provide a compact support of $\widehat{\varphi}$ whenever the assumptions of items a)-d) are not satisfied. Indeed, otherwise either $(1,0,0,0,\dots) \in \sigma_\infty$, or $(1,1,1,1,\dots) \in \sigma_\infty$, or $(1,0,1,0,\dots) \in \sigma_\infty$.

Since $(1{,}0{,}0{,}s) \notin \sigma_3$, $(1{,}0{,}1{,}s) \notin \sigma_3$, $(1{,}1{,}1{,}s) \notin \sigma_3$, for any $s = \{0{,}1\}$, as well as $(1{,}1{,}0{,}0) \notin \sigma_3$, $(1{,}1{,}0{,}1) \notin \sigma_3$, we obtain $\sigma_3 = \varnothing$, which yields $\widehat{\varphi} \in S_1^{(2)}$. Taking into account that

$$\widehat{\varphi}(1,00) = \widehat{\varphi}(1,01) = \widehat{\varphi}(1,11) = 0,$$

we get

$$\widehat{\varphi} = \mathbf{1}_{A^{-2}(U)} + b[001]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.01} + b[001]b[010]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.1} + b[001]b[011]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.11}$$

$$+b[001]b[011]b[110]\mathbf{1}_{A^{-2}(U)\,\oplus\,1.1}.$$

If now we choose $b[001] = b[011] = 1$, $|b[010]|^2 + |b[110]|^2 = 1$, then, obviously,

$$\sum\nolimits_{h\in H} |\widehat{\varphi}(h \oplus 0, ss')|^2 = 1$$

for all $s, s' \in \{0, \dots, p-1\}$. Thus, by Proposition 6, φ is orthogonal. This special case was considered in Example 1, where φ was defined by (18) with $b = b[010]$, $\beta = b[110]$. Note also, that if, moreover, $b(110) = 0$ and $b(011) = 1$, then $\varphi = \widehat{\varphi} = \mathbf{1}_U$, i.e. φ is a scaling function of the Haar basis in this case.

Example 4 Let $p = n = 3$.

(a) If $b[001] = b[002] = 0$, then $\sigma_0 = \varnothing$. Hence, $\widehat{\varphi} \in S_{-2}^{(2)}$, which yields $\widehat{\varphi} = \mathbf{1}_{A^{-2}(U)}$.
(b) If $b[001] = b[020] = b[021] = b[022] = 0$, then $\sigma_1 = \varnothing$. Hence, $\widehat{\varphi} \in S_{-1}^{(2)}$, which yields

$$\widehat{\varphi} = \mathbf{1}_{A^{-2}(U)} + \widehat{\varphi}(0,01)\mathbf{1}_{A^{-2}(U)\,\oplus\,0{,}01} = \mathbf{1}_{A^{-2}(U)} + b[001]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.01}.$$

(c) If $b[001] = b[020] = b[021] = b[022] = 0$, then similarly to (b) (changing the roles of 1 and 2), we have

$$\widehat{\varphi} = \mathbf{1}_{A^{-2}(U)} + b[002]\mathbf{1}_{A^{-2}(U)\,\oplus\,0.02}.$$

Next, following Corollary 3, one has to consider and analyse too many cases. We will restrict ourselves to only some of them. The case where $b[000]$, $b[002]$, $b[011]$, $b[012]$, $b[020]$, $b[021]$, $b[022]$, $b[201]$ and $b[210]$ were supposed to be zero, was considered in [25]. Note that if all non-zero coefficients equal 1 in absolute value, then φ is an orthogonal refinable function. Such a function φ is in $S_2^{(2)}$, i.e. $\operatorname{supp}\widehat{\varphi} \subset A^2(U)$. Our goal is to provide a greater support of $\widehat{\varphi}$.

Let $b[100] = b[101] = b[111] = b[200] = b[202] = b[222] = b[102] = b[201] = b[121] = b[122] = 0$ and $b[002]$, $b[020]$, $b[112]$, $b[221]$, $b[211]$, $b[120]$ are non-zero. Then we have $(2{,}2{,}1{,}1{,}2{,}0) \in \sigma_5$, and it is not difficult to check that $\sigma_6 = \varnothing$. Thus $\widehat{\varphi} \in S_4^{(3)} \setminus S_3^{(3)}$.

We now construct concrete refinable step functions in the case $n = 2$ for arbitrary p. Following Corollary 3, first of all we consider the case where $b[0s] = 0$ for every $s \neq 0$. Obviously, we have $\sigma_0 = \varnothing$, which together with Theorem 1 yields

m_0 is the mask of a refinable function $\varphi \in S_1^{(-1)}$, and as above, we obtain $\varphi = p^{-1}\mathbf{1}_{A(U)}$. The step is maximal possible in this case. A family of refinable step functions with all possible steps is described in the following statement (for the case $p = 3$, see Example 2).

Theorem 3 Let $m_0 \in \mathcal{M}_0^{(2)}$, $r \in \{1, \dots, p-1\}$, and let $(\xi_1, \dots, \xi_r)$ be a vector such that $\xi_k \in \{1, \dots, p-1\}$, $\xi_k \neq \xi_{k'}$ for $k \neq k'$. Suppose that $b[ss'] = 0$ whenever $s \neq 0$ and $(s, s') \neq (\xi_k, \xi_{k+1})$ for $k \in \{1, \dots, r-1\}$. Then m_0 is the mask of a refinable function $\varphi \in S_1^{(r-1)}$. Moreover, if $b[\xi_k\xi_{k+1}] \neq 0$, $k \in \{1, \dots, r-1\}$, and $b[0\xi_1] \neq 0$, then $\varphi \in S_1^{(r-1)} \setminus S_1^{(r-2)}$.

Proof Due to Theorem 1, to prove the first statement it suffices to verify that $\sigma_r = \emptyset$. If $r = 1$, then $b[ss'] = 0$ whenever $s \neq 0$, which yields $\sigma_1 = \emptyset$. Let $r > 1$. Suppose that $(s_0, \dots, s_r) \in \sigma_r$, i.e. $s_0 \neq 0$, $s_j \in \{0, 1, \dots, p-1\}$ and $b[s_{l-1}s_l] \neq 0$ for all $l \in \{0, 1, \dots, r\}$. If $s_0 = \xi_k$ for some $k \in \{1, \dots, r-1\}$, then $s_1 = \xi_{k+1}, \dots, s_{r-k} = \xi_r$. But $r - k < r$ and $b[\xi_r s] = 0$ for any digit s. Therefore, $b[s_{r-k}s_{r-k+1}] = 0$, that contradicts our assumption $(s_0, \dots, s_r) \in \sigma_r$. If $s_0 \neq \xi_k$ for all $k \in \{1, \dots, r-1\}$, then $b[s_0 s] = 0$ for any digit s, that again contradicts our assumption $(s_0, \dots, s_r) \in \sigma_r$.

To prove the second statement, we need to check that $\sigma_{r-1} \neq \emptyset$ under our additional assumptions for this case. But this holds true because $(\xi_1, \dots, \xi_r) \in \sigma_{r-1}$.

Corollary 4 Let $m_0 \in \mathcal{M}_0^{(2)}$ be the mask of a compactly supported refinable step function φ. For φ to be in $S_1^{(p-2)} \setminus S_1^{(p-3)}$, it is necessary and sufficient that there exists a vector $(\xi_1, \dots, \xi_{p-1})$, $\xi_k \in \{1, \dots, p-1\}$, such that $b[\xi_k\xi_{k+1}] \neq 0$ for all $k \in \{1, \dots, p-2\}$ and $b[0\xi_1] \neq 0$.

Proof It follows from Theorem 2 that $\sigma_{p-1} = \emptyset$, which, by Theorem 1, yields $\varphi \subset S_1^{(p-2)}$. To prove the sufficiency, it remains to repeat the final arguments of the proof of Theorem 3. To verify the necessity, we assume $\varphi \in S_1^{(p-2)} \setminus S_1^{(p-3)}$. Since $\varphi \notin S_1^{(p-3)}$, it follows from Theorem 1 that $\sigma_{p-2} \neq \emptyset$. Thus, there exists a vector $(s_0, \dots, s_{p-2})$, $s_j \in \{1, \dots, p-1\}$, $s_0 \neq \theta$, such that $b[s_{j-1}s_j] \neq 0$ for all $j \in \{1, \dots, p-2\}$ and $b[0s_0] \neq 0$. It remains to set $\xi_j = s_{j-1}$.

Theorem 4 Let m_0, r and a vector $(\xi_1, \dots, \xi_r)$ be as in Theorem 3. Suppose that $|b[\xi_k\xi_{k+1}]| = 1$ for $k \in \{1, \dots, r-1\}$, $|b[0s]| = 1$ for $s \notin \{\xi_2, \dots, \xi_r\}$ and $b[0\xi_k] = 0$ for $k \in \{2, \dots, r\}$. Then m_0 is the mask of an orthogonal refinable function $\varphi \in S_1^{(r-1)}$.

Proof By Theorem 3, m_0 is the mask of a refinable function $\varphi \in S_1^{(r-1)}$. Due to Propositions 2 and 6, to prove the orthogonality of φ, we need to check that

$$\sum_{h \in H} |\widehat{\varphi}(h \oplus s)|^2 = \sum_{h \in H_{r-1}} |\widehat{\varphi}(h \oplus s)|^2 = 1, \quad s \in \{0, \dots p-1\}, \tag{27}$$

where $H_{r-1} = H \cap A^{r-1}(U)$.

Let $k \in \{2, \dots, r\}$. By (20), if $h = \xi_1 \dots \xi_{k-1}$, then

$$|\widehat{\varphi}(h, \xi_k)| = |b[\xi_{k-1}\xi_k] \dots b[\xi_1\xi_2]b[0\xi_1]| = 1,$$

and if $h = s_1 \dots s_{k-1} \in H_{r-1}$, $h \neq \xi_1 \dots \xi_{k-1}$, then

$$\widehat{\varphi}(h, \xi_k) = b[s_{k-1}\xi_k]b[s_{k-2}s_{k-1}] \dots b[s_1 s_2]b[0s_1] = 0,$$

because for every $l = 2, \dots, r$, there exists a unique digit $s' = \xi_{l-1}$ such that $b[s'\xi_l] \neq 0$. Thus (27) is proved for $s = \xi_k$, $k = 2, \dots, r$.

Let now $s \neq \xi_2, \dots, \xi_r$. Since $b[s's] = 0$ for all $s' \neq 0$, if $h = s_1 \dots s_{k-1} \in H_{r-1}$, $h \neq \theta$, then

$$\widehat{\varphi}(h, s) = b[s_{k-1}s]b[s_{k-2}s_{k-1}] \dots b[s_1 s_2]b[0s_1] = 0,$$

and

$$|\widehat{\varphi}(0, s)| = |b[0s]| = 1,$$

which implies (27).

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Declarations

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REFERENCES

1. Schipp, F., Wade, W. R., Simon, P.: *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, New York: Adam Hilger, 1990.
2. Golubov, B., Efimov, A., Skvortsov, V.: *Walsh series and transforms. Theory and applications*, Transl. from the Russian by W. R. Wade. Mathematics and Its Applications. Soviet Series. 64. Dordrecht etc.: Kluwer Academic Publishers Group. xiii, 368 p. (1991).
3. Novikov, I. Ya., Skopina, M. A.: Why are Haar bases in various structures the same? *Math. Notes* **91** (6) (2012), 895-898.
4. Evdokimov, S., Skopina, M.: On orthogonal p-adic wavelet bases, *J. Math. Anal. Appl.*, 424 (2) (2015), 952-965.
5. Farkov, Yu. A.: Orthogonal p-wavelets on $\mathbb{R}_+$. Skopina M. (ed.), *International conference on wavelets and splines, St. Petersburg, Russia, July 3-8, 2003. Proceedings.* St. Petersburg: St. Petersburg University Press, 4-26 (2005).
6. Protasov, V. Yu., Farkov, Yu. A.: Dyadic wavelets and refinable functions on a half-line. *Sb. Math.* 97 (10) (2006), 1529-1558.
7. Novikov, I. Ya., Protassov, V. Yu., Skopina, M. A.: *Wavelet Theory*, Translations of Mathematical Monographs 239. Providence, RI: American Mathematical Society (AMS), (2011).
8. Krivoshein, A. V., Lebedeva, E. A.: Uncertainty principle for the Cantor dyadic group. *J. Math. Anal. Appl.* **423** (2) (2015), 1231-1242.
9. Platonov, S. S.: Some problems in the theory of approximation of functions on locally compact Vilenkin groups, *p-Adic Numbers Ultrametric Anal. Appl.*, 11 (2) (2019), 163-175.
10. Platonov, S. S.: Fourier transform of Dini-Lipschitz functions on locally compact Vilenkin groups, *p-Adic Numbers Ultrametric Anal. Appl.* 12 (3) (2020), 231-246.
11. Volosivets, S. S.: Embedding theorems for Hölder classes defined on p-adic linear spaces, *p-Adic Numbers Ultrametric Anal. Appl.* 12 (1) (2020), 60-67.
12. Volosivets, S. S.: Applications of p-adic generalized functions and approximations by a system of p-adic translations of a function, *Sib. Math. J.* 50 (1) (2009), 1-13.
13. Farkov, Yu. A.: Examples of frames on the Cantor dyadic group, *J. Math. Sci., New York* 187 (1) (2012), 22-34.
14. Farkov, Yu. A., Manchanda, P., Siddiqi, A. H.: *Construction of Wavelets through Walsh Functions*, Industrial and Applied Mathematics. Singapore: Springer, 2019.
15. Farkov, Yu. A.: Wavelet frames related to Walsh functions, *Eur. J. Math.* 5 (1) (2019), 250-267.
16. Farkov, Yu. A.: Frames in Walsh analysis, Hadamard matrices and uniformly distributed sets, *Proceedings of the 20th International Saratov Winter School "Modern problems of function theory and their applications"*, Saratov, January 28 – February 1, 2020 Part 1, Itogi Nauki Tekh., Ser. Sovrem. Mat. Prilozh., Temat. Obz. 199, 17-30 (2021)(Russian).
17. Protasov, V. Yu.: Approximation by dyadic wavelets, *Sb. Math.* 98 (11) (2007), 1665-1681.
18. Farkov, Yu. A., Lebedeva, E. A., Skopina, M. A.: Wavelet frames on Vilenkin groups and their approximation properties, *Intern. J. Wavelets Multiresolut. Inf. Process.*, **13** (5) (2015), 1550036 (19 pages).
19. Farkov, Yu. A.: Orthogonal wavelets with compact supports on locally compact abelian groups, *Izv. Math.* **69** (3) (2005), 623-650.
20. Hewitt, E., Ross, K. A.: *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact abelian groups*, Berlin-Heidelberg-New York: Springer-Verlag, 1970.
21. Karapetyants, M. A., Protasov, V. Yu.: Spaces of dyadic distributions, *Funct. Anal. Appl.*, 54:4 (2020), 272-277.
22. Farkov, Yu. A.: Orthogonal wavelets on direct products of cyclic groups, *Math. Notes* 82 (6) (2007), 843-859.
23. Behera, B., Jahan, Q.: *Wavelet analysis on local fields of positive characteristic*, Indian Statistical Institute Series. Singapore: Springer, 2021.
24. Debnath, L., Shah, F. A.: *Wavelet transforms and their applications*, 2nd ed. New York: Birkhäuser/Springer, 2015.
25. Lukomskii, S. F., Berdnikov, G. S.: N-Valid trees in wavelet theory on Vilenkin groups, *Intern. J. Wavelets Multiresolut. Inf. Process.* 13 (5), (2015), Article ID 1550037, 23 p.
26. Lukomskii, S. F.: Step refinable functions and orthogonal MRA on p-adic Vilenkin groups, *J. Fourier Anal. Appl.* 20 (2014), 42-65.
27. Farkov, Yu. A.: Constructions of MRA-based wavelets and frames in Walsh analysis // Poincare J. Anal. Appl. 2015, No. 2, Spec. Iss., 13-36 (2015).
28. Farkov, Yu. A.: Chrestenson-Levy system and step scaling functions, *Bulletin of the L.N. Gumilyov Eurasian National University. Ser.: Mathematics, Computer science, Mechanics.* 130 (1) (2020), 59-72.