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Soliton Solutions to Hydrodynamic Equations When Describing Collisions and Oscillations of Atomic Nuclei

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Abstract—A hydrodynamic approach is used to find an analytical solution to hydrodynamic equations in a soliton approximation for one- and two-dimensional layer collisions. The stages of compression, decompression, and expansion are investigated using a single formula for layers with energies of around 10 MeV per nucleon. Two-dimensional generalization produces a region of a rarefied bubble at the stage of expansion. The approach is of intrinsic interest and can be used in other fields of physics to calculate the nonlinear dynamics of oscillations of complex systems.

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INTRODUCTION

The hydrodynamics of interaction between complex systems is considered in this work using the example of an atomic nucleus consisting of nucleons. The properties of the nucleus can be described using the liquid droplet model proposed by Bohr and Wheeler for the fission process [1]. Stöcker and Greiner applied hydrodynamics to heavy ion collisions in [2].

The choice of the equation of state (EOS) is important when studying heavy ion collisions and different impacts. In the hydrodynamic approach, a local equilibrium equation of state corresponding to the onset of local thermodynamic equilibrium can be chosen as the EOS [2]. We proposed a nonequilibrium EOS in [3–10].

Solutions to these hydrodynamic equations in a one-dimensional case was found analytically using single-soliton solutions to the Korteweg–de Vries equation [11, 12] for both the weak nonlinearity in [13] and high-amplitude nonlinear perturbations of shock waves. The stages of compression, decompression, and expansion of a substance were described using a single formula. This is of independent interest, since solitons play an important role in elementary particle physics, nuclear physics, and general physics. It was important for us to ensure that the dispersion terms that appear in the hydrodynamic and Korteweg–de Vries equation do not violate the concept of the formation of a hot spot. Generalizing this idea to a two-dimensional case yields similar dynamics of the oscillations of a complex system and a region of rarefaction: a bubble at the center of an expanded system. This can be extended to a wide range of engineering applications.

The formation and existence of bubble nuclei is a nuclear oddity. The search for and study of vesicular nuclei has a long history. It was studied by, e.g., Siemens and Bethe in [14] and Wong in [15]. The possibility of a stable bubble forming was substantiated in [16] only for superheavy nuclei with charge number $Z > 120$, using a droplet model with shell corrections. However, rarefaction at the center was predicted for the proton density of doubly magic nucleus ^{34}Si using the Hartree–Fock–Bogoliubov formula and confirmed experimentally in [17]. The relativistic Hartree–Fock–Bogoliubov model was used in [18] to prove the existence of a bubble in the ^{48}Si nucleus for the densities of both neutrons and protons at the center of the latter. The above are static solutions for bubbles. In our calculations, a dynamic bubble always appears in the region of rarefaction at a hydrodynamic system's stage of expansion.

HYDRODYNAMICS EQUATIONS

We used the kinetic equation in [4–8] to find nucleon distribution function $f(\vec{r},\vec{p},t)$ (where $\vec{r}(x_1,x_2,x_3)$ is a spatial coordinate, $\vec{p}(p_1,p_2,p_3)$ is a pulse, and t is time). This can be extended to arbitrary dynamical systems. The solution to the kinetic equation for distribution function $f(\vec{r},\vec{p},t)$ is sought in the form

$$f(\vec{r},\vec{p},t) = f_1 q + f_0(1-q), \tag{1}$$

where $f_0(\vec{r},\vec{p},t)$ and $f_1(\vec{r},\vec{p},t)$ are the functions of the local equilibrium distribution and the nonequilibrium distribution, respectively.

With nonequilibrium at $q = 1$, we obtain equations of long-range hydrodynamics [3] for a one-dimensional case:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\upsilon)}{\partial x} = 0, \quad (2)$$

$$\frac{\partial(m\rho\upsilon)}{\partial t} + \frac{\partial\left(m\rho\upsilon^2 + P\right)}{\partial x} = 0, \quad (3)$$

$$\frac{\partial\left(e + m\rho\upsilon^2/2\right)}{\partial t} + \frac{\partial\left(\upsilon\left(e + m\rho\upsilon^2/2 + P\right)\right)}{\partial x} = 0. \quad (4)$$

This system of nonlinear partial differential equations is normally calculated numerically. In this work, we develop an approach for finding approximate analytical solutions to these equations in cases of weak nonlinearity (by reducing them to the Korteweg–de Vries equations) and high-amplitude perturbations (using soliton-like solutions). This approach is generalized to a two-dimensional case at density $\rho(x,t)$, which depends only on coordinate x and time t. This had never been considered before and can be extended to both atomic nuclei and arbitrary complex systems.

ANALYTICAL SOLUTION TO HYDRODYNAMIC EQUATIONS USING SOLITONS

In a one-dimensional case, hydrodynamic equations with relaxation factor $q = 1$ (a nonequilibrium case) are reduced to system of Eqs. (2)–(4) for finding the nucleon density $\rho(x,t)$, velocity $\upsilon(x,t)$, and heat energy density $I(x,t)$.

It follows from Eqs. (2) and (4) that the heat term is $I = I_1\left(\frac{\rho}{\rho_0}\right)^3$, where I_1 is a coefficient independent of ρ. We seek a joint solution of Eqs. (2) and (3) in the form $\upsilon = \upsilon(\rho)$ and obtain two Korteweg–de Vries equations [13].

In other words, hydrodynamic equations can be reduced to two Korteweg–de Vries equations. This allows us to describe a collision between complex systems (nuclei) as a collision of solitons, if the simple wave of the Korteweg–de Vries equation is integrated over x_1. We therefore find

$$Z = \int_0^L \varsigma \frac{dx_1}{L}, \quad (5)$$

where L is the thickness of a layer, Z is a simple Korteweg–de Vries wave emitted by the layer, and $\zeta(x - x_1, t)$ is the one-soliton solution to the Korteweg–de Vries equation. This applies to each nuclear layer as a source of simple waves. By allowing for multiple reflections of the Korteweg–de Vries waves from the system's interfaces, we can consider the entire dynamics of the collision of nuclear slabs [13].

Let us now consider the propagation of perturbations of arbitrary amplitude. For the energy density, we can use the simple expression

$$e = K(\rho - \rho_0)^2, \quad (6)$$

where $\rho_0 = 0.15$ fm^{-3} is the equilibrium density and K is the pressure modulus. The pressure is then

$$P = -\frac{\partial(e/\rho)}{\partial(1/\rho)} = K\left(\rho^2 - \rho_0^2\right) - \alpha\left(\frac{\partial\rho}{\partial x}\right)^2. \quad (7)$$

Here we add a dispersion term with coefficient α, where $\frac{\alpha}{2mc_{so}^2}\rho_0 = (\text{fm})^2$ and the speed of sound is $c_{s0} \approx 1/3c \approx 10^8$ m/s. In the event of a collision between two nuclei, shock waves form that propagate at velocity D, which can be found using Eqs. (2) and (3), assuming that $\frac{\partial}{\partial t} = -D\frac{\partial}{\partial x}$. Integrating these equations over the jump in density, we obtain

$$D = -\frac{\rho_0\upsilon_0}{\rho - \rho_0}, \quad (8)$$

where υ_0 is the initial velocity of colliding nuclei. Assuming velocity D to be equal to speed of sound $c_s = \sqrt{\frac{\partial P}{m\partial\rho}}$, and in light of pressure (7), we obtain the equation for density ρ:

$$\pm\frac{(\rho - \rho_0)}{(\rho_1 - \rho)}\frac{d\rho}{dx}\sqrt{\alpha} = \sqrt{K(2\rho_1(\rho - \rho_0) + (\rho - \rho_0)^2)}, \quad (9)$$

where ρ_1 is the maximum density of compression on a shock wave,

$$2K\rho_1 = \frac{(\rho_0\upsilon_0)^2}{(\rho_1 - \rho_0)^2}. \quad (10)$$

Equation (9) can be integrated implicitly in elementary functions, but the answer is rather cumbersome. Separating the main terms of the solution, we arrive at

$$\frac{(\rho - \rho_1)(\rho - \rho_0)}{\rho_1^2} = -\exp\left(\pm\sqrt{\frac{K}{\alpha}}x\right), \quad (11)$$

where the plus sign corresponds to the solution near ρ_1, and the minus sign corresponds to the one near ρ_0. The solutions must be sewn together at an intermediate value at the inflection point, in order to obtain the solution to a kink for a propagating wave. The reasoning for negative values of x is similar, but since we are not interested now in details of the wave front struc-

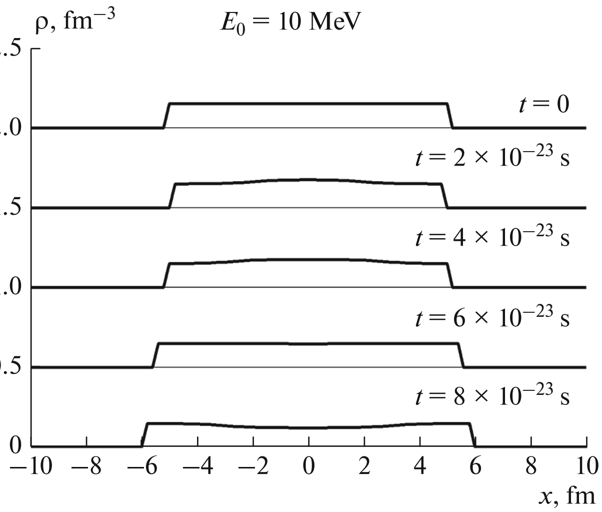

Fig. 1. Instantaneous profiles of collisions beteen nuclear slab layers at energy $E_0 = 10$ MeV per nucleon at times t = 2, 4, 6, and 8 in units of 10^{-23} s.

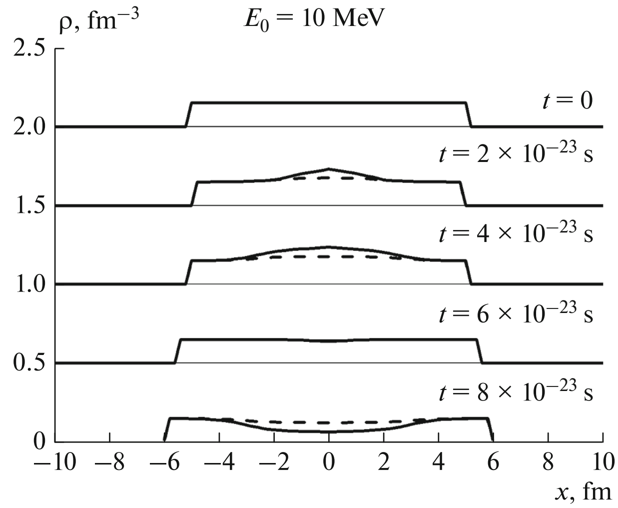

Fig. 2. Instantaneous profiles of collisions of identical nuclei (solid lines) at energy $E_0 = 10$ MeV per nucleon at times t = 2, 4, 6, and 8 in units of 10^{-23} s for a two-dimensional case. As in Fig. 1, dashed lines show the density profiles of one-dimensional layers.

ture, we can approximate solution (11) with the soliton solution

$$\rho = \rho_0 + 4\frac{(\rho_1 - \rho_0)}{(\exp(-\lambda x/2) + \exp(\lambda x/2))^2}, \qquad (12)$$

where $\lambda = \sqrt{\frac{K}{\alpha}}$. Equation (12) describes the main features of solution (11).

As we did earlier with the Korteweg–de Vries solitons, we can integrate Eq. (12) over the length of the layer and consider the propagation of a shock wave front and its reflection off interfaces. Integration yields

$$\rho = \frac{1}{L}\int_{l_1}^{l_2} \rho' \, dx = \rho_0 + 4\frac{(\rho_1 - \rho_0)}{\lambda L} \times \left[\frac{1}{1 + \exp(\lambda(x - l_2 - Dt))} - \frac{1}{1 + \exp(\lambda(x - l_1 - Dt))}\right], \qquad (13)$$

where ρ' is formula (12), l_1 and l_2 are the boundaries of the nucleus, and $L = l_2 - l_1$ is its size. Since we have velocity $\upsilon = 0$ at the maximum of the shock wave, Eqs. (2) and (3) yield a wave equation for the maximum that allows a d'Alembert solution, which is what we used. Velocity υ can also be found from Eq. (2) by using Eq. (13) for density, allowing for possible reflections of shock waves from the boundaries of a system and the motion of the boundaries.

We can therefore study the full dynamics of nuclear collisions in a one-dimensional case using soliton solutions (12) and (13).

Figure 1 shows the density profiles for collisions between identical nuclei with size L = 5 fm and energy $E_0 = 10$ MeV per nucleon at times t = 2, 4, 6, and 8 in units of 10^{-23} s. We can see the initial compression, formation of a hot spot, and subsequent expansion of nuclei with the rise of rarefaction in the central region.

We also found a simplified solution to the problem in a two-dimensional case. The equations were obtained from hydrodynamic equations by integrating the latter over the transverse coordinate by assuming that density $\rho(x,t)$ was independent of coordinate y. The solution was obtained by using formula (12), replacing $\rho_1 \to \rho_1 S(x,y,t)$ and then dividing any perturbation by S, where $S(x,y,t) = (y_0(x) + \upsilon_y t)^2 / y_0^2$, and υ_y coincides with the speed of sound.

Figure 2 shows density profiles for a collision between identical nuclei with longitudinal size L = 5 fm and energy $E_0 = 10$ MeV per nucleon at times t = 2, 4, 6, and 8 in units of 10^{-23} s. These results are shown by solid lines. Dashed lines correspond to a one-dimensional case. We can see the oscillations of compression and rarefaction are stronger in a two-dimensional case.

Figure 3 shows the dependence of transverse size $y = y_0(x) + \upsilon_y t$ on coordinate x at times t = 2, 4, 6, and 8 in units of 10^{-23} s. These results were obtained for a density independent of coordinate y. Figure 2 shows that a rarefied region forms in the center of the nucleus at the end of the collision, and normal density is observed at the ends. We would therefore expect the formation of a bubble at the center of the nucleus.

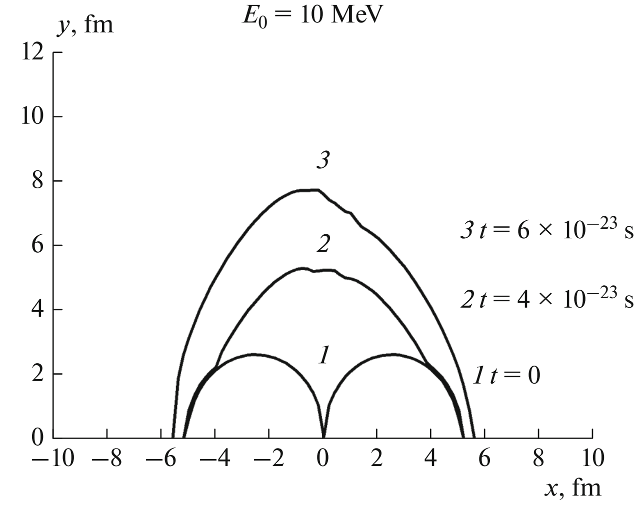

Fig. 3. Profiles of the maximum transverse size of nuclei at energy $E_0 = 10$ MeV per nucleon at times $t = 2, 4, 6$, and 8 in units of 10^{-23} s.

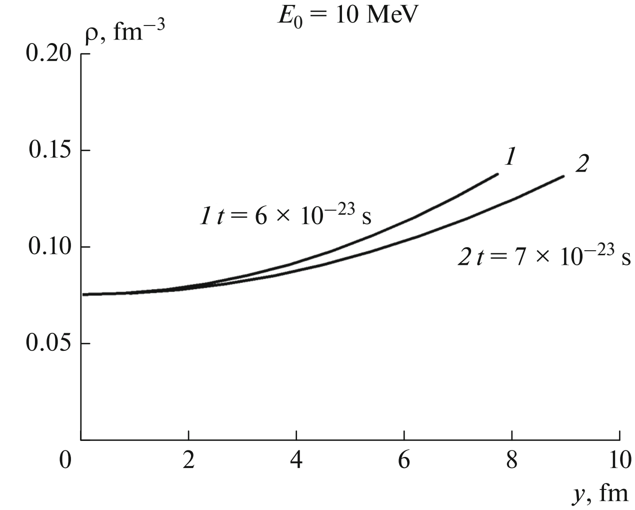

Fig. 4. Dependence of density $\rho(y,t)$ on transverse coordinate y for $x = 0$ at energy $E_0 = 10$ MeV per nucleon at times $t = 6, 7$ in units of 10^{-23} s.

The situation is similar for a transverse coordinate. Figure 4 shows the change in density $\rho(y) = \rho_0'\left(1 + \frac{(y - y_0)^2}{4t^2c_s^2}\right)$ at $x = 0$ and different moments in time. This dependence is obtained by solving the Euler hydrodynamic equation for velocity $\upsilon_y = \frac{y - y_0}{t}$ with regard to the correction for the density change with coordinate y. Our estimate for a two-dimensional case confirms the formation of a bubble at the center of the nucleus, where we observe the formation of a rarefied region with density ρ_0'. At low energies of colliding systems, the oscillations in density for alternating rarefactions and compressions can produce a stable bubble at the center of the nucleus.

CONCLUSIONS

A nonequilibrium hydrodynamic approach to describing complex systems was developed using the example of a collision between atomic nuclei. The nonequilibrium approach to hydrodynamic equations allowed us to describe experimental data better than the equation of state corresponding to conventional hydrodynamics, which assumes the establishment of the local thermodynamic equilibrium. In this description, was essential to identify a hot spot. It was shown that introducing dispersion terms does not violate this representation. At the stage of expansion, a rarefied region (a dynamic bubble) formed at the center of the system. Our investigation was performed for both one- and two-dimensional cases.

Our hydrodynamic equations and the Korteweg–de Vries equation were of the same nature, and the resulting soliton-like solutions could be applied to any complex system. Reducing the equations of hydrodynamics to the solution to two Korteweg–de Vries equations in the form of solitons, we can find an analytical solution to the problem. Generalizing the results from our soliton approach to high-amplitude perturbations and extra dimensions is of independent fundamental interest to general physics and can be used in different fields of engineering.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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