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RINGS GENERATED BY CONVERGENCE SETS OF MULTIDIMENSIONAL COMPLETE FIELD

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The convergence sets of a multidimensional complete field are those having the property that all power series over it converge when substituting an element of the maximal ideal for a variable. It is proved that a convergence set lies in the ring of integers if and only if it is contained in some convergence ring. Bibliography: 10 titles.

1. Basic concepts

Multidimensional local fields (see [6,7]) form a natural generalization of classical local fields.

We call a finite field 0-dimensional local. A complete discrete valuation field $K = K_n$ is n-dimensional local ($n \geq 1$) if its residue field K_{n-1} is an $(n-1)$-dimensional local field. Since many properties are preserved if 0-dimensional field is replaced by arbitrary perfect field, we will also consider this case and call K a multidimensional complete field.

Let us recall the basic concepts related with multidimensional local and complete fields (see [2]).

Let t_n be an uniformizing element of the field K_n with respect to a valuation v_n, and let t_{n-1} be a unit in K_n, the residue class $\overline{t_{n-1}}$ of which is a uniformizing element of K_{n-1}, and so on up to a unit t_1 in K_n, K_{n-1}, …, K_2, the residue class of which in K_1 becomes a uniformizing element. The set $\bar{t} = (t_1, \ldots, t_n)$ is called the system of local parameters of K and defines the valuation $\bar{v} = (v^{(1)}, \ldots, v^{(n)})$ given by the formulas

$$v^{(k)}(a) = v_{K_k}\left(\overline{a t_n^{-v^{(n)}(a)} \ldots t_{k+1}^{-v^{(k+1)}(a)}}\right),$$

where $a \neq 0$ and $\bar{v}(0) = +\infty$.

In what follows, we put

$$\bar{r}_k = (r_{k+1}, \ldots, r_n) \ (0 \leq k \leq n-1), \quad \bar{r}_0 = \bar{r}, \quad \bar{t}_k^{\bar{r}_k} = t_{k+1}^{r_{k+1}} \ldots t_n^{r_n}.$$

Moreover, we use a lexicographic ordering on $\mathbb{Z}^n$, namely, $\bar{r}^{(1)} < \bar{r}^{(2)}$ if $\bar{r}_k^{(1)} = \bar{r}_k^{(2)}$ and $r_k^{(1)} < r_k^{(2)}$ for some $1 \leq k \leq n$.

The set $\mathcal{O} = \{a \in K : \bar{v}(a) \geq 0\}$ forms a valuation ring independent of the choice of local parameters, the unique maximal ideal of which is $\wp = \{ a \in \mathcal{O} : \bar{v}(a) > 0 \}$. The group of units of a ring is denoted by U

The most important tool for studying multidimensional local and complete fields is the following structure theorem (see [1,6]).

Theorem 1.1. *Let $K = K^{(n)}$ be an n-dimensional complete field.*
1. *If* $\operatorname{char} K = \operatorname{char} K_0$, *then* $K \approx K_0((t_1)) \ldots ((t_n))$.
2. *For* $\operatorname{char} K = 0$ *and* $\operatorname{char} K_0 = p > 0$, *let* $F_0 = \operatorname{Frac}(W(K_0))$ *be the field of quotients of the ring of Witt vectors over the last field of residues.*

If $\operatorname{char} K^{(s)} = 0$ *and* $\operatorname{char} K^{(s-1)} = p > 0$, $2 \leq s \leq n$, *then K is a finite completely ramified extension of the field*

$$F\{\{t_1\}\} \ldots \{\{t_{s-1}\}\}((t_{s+1})) \ldots ((t_n)),$$

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where F is a finite completely ramified extension of F_0. Moreover, K has a finite extension of the form

$$L\{\{T_1\}\}\dots\{\{T_{s-1}\}\}((T_{s+1}))\dots((T_n)),$$

obtained by adding an element algebraic over F_0, where L is a finite extension of F_0.
3. *If* $\operatorname{char} K^{(1)} = 0$ *and* $\operatorname{char} K^{(0)} = p$, *then* $K \approx k((t_2))\dots((t_n))$, *where* $k = K^{(1)}$ *is a finite completely ramified extension of* F_0.

Here, for a field F complete with respect to a discrete valuation w, we mean that

$$F\{\{t\}\} = \left\{ \sum_{i=-\infty}^{\infty} c_i t^i \,:\, c_i \in F,\ w(c_i) \geq c > -\infty,\ \lim_{i\to-\infty} w(c_i) = +\infty \right\}$$

with valuation $v\left(\sum\limits_{i=-\infty}^{\infty} c_i t^i\right) = \min_i w(c_i)$. It is easily seen that it is a complete discrete valuation field with residue field $\bar{F}((\bar{t}))$.

2. Convergence sets

Definition 2.1. A number $s = s(K)$ such that

$$\operatorname{char} K^{(s)} = 0, \quad \operatorname{char} K^{(s-1)} = p > 0,$$

is called the inertial number of the field K. We put $s(K) = 1$ if

$$\operatorname{char} K^{(1)} = 0, \quad \operatorname{char} K^{(0)} = p,$$

and $s(K) = 0$ if $\operatorname{char} K = \operatorname{char} K_0$.

Thus, the inertial number $s(K)$ is the last number in numbering the residue fields from n to zero, for which the characteristic of the original field $K = K_n$ is preserved.

Let us introduce the following notion (see [4]).

Definition 2.2. A set of multi-indices $\Omega \subset \mathbb{Z}^n$ is called an admissible set if for any $\bar{i}_k \in \mathbb{Z}^{n-k} (1 \leq k \leq n)$ there exists $I(\bar{i}_k) = I \in \mathbb{Z}$ such that $\bar{r}_k = \bar{i}_k$ implies $r_k \geq I$ for all $\bar{r} \in \Omega$. The quantities

$$\omega_k(\bar{i}_k) = \sup I(\bar{i}_k)$$

are called the characteristic indices of the admissible set.

In fact, $\omega_k(\bar{r}_k)$ is the smallest value of r_k, for which Ω contains a multi-index ending in $\bar{r}_k$ (otherwise, $\omega_k(\bar{r}_k) = +\infty$).

We choose a complete system B of representatives of nonzero elements K_0 in K, and a system $\bar{t}$ of local parameters. Then (see [2, 5]) any nonzero element of K can be represented as the series

$$a = \sum_{\bar{r}\in\Omega^a} a_{\bar{r}} \bar{t}^{\bar{r}}, \quad a_{\bar{r}} \in B,$$

where Ω^a is an admissible set and B can be chosen so that this representation is unique. Moreover,

$$a = \sum_{\bar{r}_k\in\Omega^a_k} a_{\bar{r}_k} \bar{t}_k^{\bar{r}_k}, \quad 0 \leq k \leq n-1,$$

where $\Omega^a_k \subset \mathbb{Z}^{n-k}$ is an admissible set.

Denoting by $\omega_k^a(\bar{r}_k)$ the characteristic indices of a, we have

$$a \in \mathcal{O} \Leftrightarrow \omega_k^a(\bar{0}_k) \geq 0, \quad k = \overline{n;1},$$
$$a \in \wp \Leftrightarrow \omega_k^a(\bar{0}_k) \geq 0, \quad k = \overline{n;2}, \quad \omega_1^{(a)}(\bar{0}_1) \geq 1,$$
$$a \in U \Leftrightarrow \omega_k^a(\bar{0}_k) = 0, \quad k = \overline{n;1}.$$

It is easily seen that $a_{\omega_k(\bar{r}_k)\bar{r}_k} \neq 0$ and the index $r_k = \omega_k^a(\bar{r}_k)$ is the smallest of those having this property, and $\omega_k^a(\bar{r}_k) = +\infty$ if and only if the coefficient of $\bar{t}_k^{\bar{r}_k}$ is zero.

Let Ω^a and Ω^b be admissible sets of the elements $a, b \in K$. Then (see [5])

$$\omega_k^{a+b}(\bar{r}_k) \geq \inf_{\bar{i}_k \leq \bar{r}_k} (\omega_k^a(\bar{i}_k), \omega_k^b(\bar{i}_k)),$$
$$\omega_k^{ab}(\bar{r}_k) \geq \inf_{\bar{i}_k + \bar{j}_k \leq \bar{r}_k} (\omega_k^a(\bar{i}_k) + \omega_k^b(\bar{j}_k)) \quad (k = \overline{n;1})$$

(if $k = n$, then the corresponding argument and condition on indices are absent).

Moreover, if at least one of $a_{\bar{r}_k}, b_{\bar{r}_k}$ is nonzero, then $\omega_k^{a+b}(\bar{r}_k) \geq \inf(\omega_k^a(\bar{r}_k), \omega_k^b(\bar{r}_k))$, and if $\omega_k^a(\bar{r}_k) \neq \omega_k^b(\bar{r}_k)$, then $\omega_k^{a+b}(\bar{r}_k) = \inf(\omega_k^a(\bar{r}_k), \omega_k^b(\bar{r}_k))$.

In the case of standard field, these formulas can be refined.

Let $s = s(K)$ be the inertial number. Then for $k < s$, we have

$$\omega_k^{a+b}(r_{k+1}, \ldots, r_s, \ldots, r_n)$$
$$\geq \inf_{i_s \leq r_s} (\omega_k^a(r_{k+1}, \ldots, i_s, \ldots, r_n), \omega_k^b(r_{k+1}, \ldots, i_s, \ldots, r_n)),$$
$$\omega_k^{ab}(\bar{r}_k) \geq \inf_{\substack{i_l + j_l = r_k, l \neq s \\ i_s + j_s \leq r_s}} (\omega_k^a(\bar{i}_k) + \omega_k^b(\bar{j}_k)),$$

and if $k \geq s$, then

$$\omega_k^{a+b}(\bar{r}_k) \geq \inf(\omega_k^a(\bar{r}_k), \omega_k^b(\bar{r}_k)),$$
$$\omega_k^{ab}(\bar{r}_k) \geq \inf_{\bar{i}_k + \bar{j}_k = \bar{r}_k} (\omega_k^a(\bar{i}_k) + \omega_k^b(\bar{j}_k)).$$

For multidimensional fields, the topology of discrete valuation (in which the defining elements of a multidimensional field can diverge) is less convenient than the so-called Parshin topology (see [8]), in which all these series converge. It is defined recursively using topologies of residue fields (see [2, 3]).

For finite extensions of multidimensional complete fields, the Parshin topology of subfield coincides with the topology induced by a superfield, and the topology of the subfield extends uniquely to the superfield. By the structure theorem, this reduces questions of convergence to the standard fields $F\{\{t_1\}\} \ldots \{\{t_{s-1}\}\}((t_{s+1})) \ldots ((t_n))$ (we also include here the degenerate cases $s(K) = 1$ and $s(K) = 0$).

Some of the results related with various variants of the Hilbert symbol can be generalized to multidimensional fields (see [9, 10]) . In this case, power series arise with coefficients from the ring of integers, the value of which is calculated by substituting an element of the maximal ideal instead of the variable. In the Parshin topology, these series can diverge. Since the recursive definition of the topology is inconvenient for practical use, convergence criteria based on the explicit properties of the elements are needed.

In [5], a criterion for an infinitesimal and a sufficient condition for the convergence of a series are proved.

Theorem 2.1. *A sequence $\{a^{(m)}\}_{m \geq 1}$ is infinitesimal if and only if*

1. *for* $k=\overline{n;2}$ *and all* $\bar r_k$,

$$R_k(\bar r_k)=\inf_m(\omega_k^{a^{(m)}}(\bar r_k))>-\infty,$$

2. *for all* $\bar r_1$,

$$R_1(\bar r_1)=\lim_{m\to+\infty}\omega_1^{a^{(m)}}(\bar r_1)=+\infty.$$

Theorem 2.2. *Let* $c(X)=\sum\limits_{m\ge 0}c^{(m)}X^m\in K[[X]], x\in K$. *The following set of conditions is sufficient for the convergence of* $c(X)$ *to* $c(x)$ *under substituting* x *instead of* X*:*
1. *for* $k=\overline{n;2}$ *and all* $\bar r_k$,

$$R_k(\bar r_k)=\inf_m\inf_{\bar i_k+\bar i_k^{(1)}+\dots+\bar i_k^{(m)}\le\bar r_k}(\omega_k^{c^{(m)}}(\bar i_k)+\omega_k^{x}(\bar i_k^{(1)})+\dots+\omega_k^{x}(\bar i_k^{(m)}))>-\infty,$$

2. *for all* $\bar r_1$,

$$R_1(\bar r_1)=\lim_{m\to+\infty}\inf_{\bar i_1+\bar i_1^{(1)}+\dots+\bar i_1^{(m)}\le\bar r_1}(\omega_1^{c^{(m)}}(\bar i_1)+\omega_1^{x}(\bar i_1^{(1)})+\dots+\omega_1^{x}(\bar i_1^{(m)}))=+\infty.$$

In [4], a convenient notion of a convergence set is introduced and a criterion for such a set is proved.

Definition 2.3. A set $A\subset K$ is called a convergence set if any power series $c(X)=\sum\limits_{m\ge 1}c^{(m)}X^m\in A[[X]]$ converges when substituting for X an arbitrary element of the maximal ideal $\wp$.

Theorem 2.3. $A\subset K$ *a convergence set if and only if for* $k=\overline{n;1}$ *and all* $\bar r_k$,

$$\omega_k^A(\bar r_k)=\inf_{a\in A}\omega_k^a(\bar r_k)>-\infty.$$

The quantities $\omega_k^A(\bar r_k)$ are called the characteristic indices of the convergence set A.

3. Rings generated by convergence set

In [10], subrings of the ring of integers of a multidimensional complete field are constructed that are convergence rings, and formal Lubin–Tate groups are constructed over these rings. However, if the convergence set is not a ring, the formal groups over it are inconvenient.

Let us ask the following question. Let K be a standard field and A a convergence set. Whether finite sums and products of A belong to some (not necessarily the same) convergence set?

For arbitrary $a_1,\dots,a_q\in A$, we have a relation for the characteristic indices of their sum: for $k\ge s$,

$$\omega_k^{a_1+\dots+a_q}(\bar r_k)\ge\inf_{1\le\alpha\le q}(\omega_k^{a_\alpha}(\bar r_k))\ge\omega_k^A(\bar r_k),$$

and if $k<s$, then

$$\omega_k^{a_1+\dots+a_q}(r_{k+1},\dots,r_s,\dots,r_n)\ge\inf_{\omega_s^A(\bar r_s)\le i_s\le r_s}(\omega_k^A(r_{k+1},\dots,i_s,\dots,r_n)).$$

Note that the values limiting the indices from below do not depend on the terms themselves or on their number, that is, all finite sums of elements of the convergence set A belong to a certain convergence set which is a group with respect to addition. As such set, we can take all possible series $\widetilde A$ of the form

$$\sum_{\bar r\in\widetilde\Omega}a_{\bar r}\bar t^{\bar r},\quad a_{\bar r}\in B\cup\{0\},$$

where B is a complete system of representatives of nonzero elements K_0 in K, and the admissible collection $\widetilde{\Omega}$ consists of all multiindices satisfying the conditions

$$r_k \geq \omega_k^{\widetilde{A}}(\bar{r}_k) = \inf_{\omega_s^A(\bar{r}_s) \leq i_s \leq r_s} (\omega_k^A(r_{k+1}, \dots, i_s, \dots, r_n))$$

if $k < s$, and $r_k \geq \omega_k^{\widetilde{A}}(\bar{r}_k) = \omega_k^A(\bar{r}_k)$ if $k \geq s$.

Thus, finite sums of elements of an arbitrary convergence set lie in some (in general, different) convergence set.

Clearly, this is not the case for the product: if $A \not\subset \mathcal{O}$ and hence there is $a \in A$ such that $\omega_k^a(\overline{0}_k) < 0$ for some k, its degrees do not lie in any convergence set.

As shown in [9], any convergence set is represented as $A = \bar{T}^{\bar{R}}G$, where G is a convergence set lying in the ring of integers (that is, $\omega_k^G(\overline{0}_k) \geq 0$, $k = \overline{n;1}$).

In what follows, G is a convergence set of this type. Then, obviously, $G \subset \widetilde{G} \subset \mathcal{O}$.

The relation between the characteristic indices of the product of the elements $a_1, \dots, a_q \in G$ has the form

$$\omega_k^{a_1 \dots a_q}(\bar{r}_k) \geq \inf_{\substack{k+1 \leq l \leq n \\ i_l^{(1)} + \dots + i_l^{(q)} = r_l, l \neq s \\ i_s^{(1)} + \dots + i_s^{(q)} \leq r_s}} (\omega_k^{a_1}(\bar{i}_k^{(1)}) + \dots + \omega_k^{a_q}(\bar{i}_k^{(q)}))$$

(if $k \geq s$, then the inequality $i_s^{(1)} + \dots + i_s^{(q)} \leq r_s$ does not affect the result).

By the condition, $i_l^{(\alpha)} \geq \omega_l^G(\bar{i}_l^{\alpha})$. Obviously, $\omega_n^{a_1 \dots a_q} \geq 0$. Put $\widehat{\omega}_n^G = 0$.

Denote by q_n the number of nonzero indices i_n over which the enumeration is carried out and number them first. Since $i_n^{(1)} + \dots + i_n^{(q)} \leq r_n$ and $\omega_n^G \geq 0$, we have $0 \leq q_n \leq r_n$, and also the indices satisfy the conditions $1 \leq i_n^{(\alpha)} \leq r_n$ and $i_{n-1}^{\alpha} \geq \inf_{1 \leq i_n \leq r_n} \omega_{n-1}(i_n) = \widetilde{\omega}_{n-1}(r_n)$ for $1 \leq \alpha \leq q_n$. If $\alpha > q_n$, then $i_n^{(\alpha)} = 0, i_{n-1}^{(\alpha)} \geq \omega_{n-1}(0) \geq 0$.

Consequently,

$$\begin{aligned} \omega_{n-1}^{a_1 \dots a_q}(r_n) &\geq \inf_{i_n^{(1)} + \dots + i_n^{(q)} \leq r_n} (\omega_{n-1}^{a_1}(i_n^{(1)}) + \dots + \omega_{n-1}^{a_{n-1}}(i_n^{(q)})) \\ &\geq q_n \widetilde{\omega}(r_n) \geq r_n \inf(0, \widetilde{\omega}_{n-1}(r_n)). \end{aligned}$$

Thus if $\widehat{\omega}_{n-1}^G(r_n) = r_n \inf(0, \widetilde{\omega}_{n-1}(r_n))$, then $\omega_{n-1}^{a_1 \dots a_q}(r_n) \geq \widehat{\omega}_{n-1}^G(r_n)$.

Next,

$$\omega_{n-2}^{a_1 \dots a_q}(r_{n-1}, r_n) \geq \inf_{\substack{i_n^{(1)} + \dots + i_n^{(q)} \leq r_n \\ i_{n-1}^{(1)} + \dots + i_{n-1}^{(q)} \leq r_{n-1}}} (\omega_{n-2}^{a_1}(i_{n-1}^{(1)}, i_n^{(1)}) + \dots + \omega_{n-2}^{a_{n-1}}(i_{n-1}^{(q)}, i_n^{(q)})).$$

We know that $\widehat{\omega}_{n-1}^G(r_n) \leq i_{n-1}^{(1)} + \dots + i_{n-1}^{(q)} \leq r_{n-1}$. Then

$$\inf(0, \widetilde{\omega}_{n-1}(r_n)) \leq i_{n-1}^{(\alpha)} \leq r_{n-1} - \widehat{\omega}_{n-1}^G(r_n), \quad 1 \leq \alpha \leq q.$$

Moreover if q_{n-1} is the number of nonzero (i.e., positive) indices $i_{n-1}^{(\alpha)}$ corresponding to $i_n^{(\alpha)}$ equal to zero, then

$$q_{n-1} \leq r_{n-1} - \widehat{\omega}_{n-1}^G(r_n).$$

Thus the number of nonzero pairs of indices over which the estimate is taken does not exceed some number depending on r_{n-1}, r_n only, and does not depend on $a_1, \dots, a_q$. Moreover the variants of nonzero pairs also do not depend on the choice of factors. Since $\omega_{n-2}(0,0) \geq 0$, we get

$$\omega_{n-2}^{a_1 \dots a_q}(r_{n-1}, r_n) \geq \widehat{\omega}_{n-2}^G(r_{n-1}, r_n) > -\infty.$$

Continuing the argument, we see that each time there is a finite number of nonzero multi-indices, independent of q and $a_1, \dots, a_q$, which makes it possible to take the previous index negative. This means that then a finite set of nonzero indices independent of q and $a_1, \dots, a_q$ is added, that is, $\omega_k^{a_1 \dots a_q}(\bar{r}_k) \geq \widehat{\omega}_k^G(\bar{r}_k)$ for all $\bar{r}_k$.

Thus given any convergence set $G \subset \mathcal{O}$, we have constructed a convergence ring $\widehat{G}$ such that $G \subset \widehat{G} \subset \mathcal{O}$.

Definition 3.1. Let A be a convergence set. The smallest with respect to inclusion convergence ring containing A is called the convergence ring generated by A. It is denoted by $\widehat{A}$.

We have proved the following criterion.

Theorem 3.1. *Let A be a convergence set. It generates a convergence ring $\widehat{A}$ if and only if $A \subset \mathcal{O}$. In this case, $\widehat{A} \subset \mathcal{O}$.*

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REFERENCES

1. I. B. Zhukov, "Structural theorem for full fields," *Trudy St. Peterburg. Mat. Ob.*, **3**, 215–234 (1994).
2. I. B. Zhukov and A. I. Madunts, "Multidimensional complete fields: topology and other basic constructions," *Trudy St. Peterburg. Mat. Ob.*, **3**, 186–196 (1994).
3. I. B. Zhukov and A. I. Madunts, "Additive and multiplicative expansions in higher-dimensional local fields," *Zap. Nauchn. Semin. POMI*, **272**, 186–196 (2000).
4. A. I. Madunts, "Convergence sets of a multidimensional complete field," *Zap. Nauchn. Semin. POMI*, **492**, 125–133 (2020).
5. A. I. Madunts, "On convergence of sequences and series in multidimensional complete fields," Ph.D. Thesis, St.Petersburg, 1–14 (1995).
6. A. N. Parshin, "Abelian coverings of arithmetic schemes," *Sov. Math. Dokl.*, **19**, 1438–1442 (1978).
7. A. N. Parshin, "On the arithmetic of two-dimensional schemes. I. Distributions and residues," *Math. USSR-Izv.*, **10**, No. 4, 695–729 (1976).
8. A. N. Parshin, "Local class field theory," *Tr. MIAN SSSR*, **165**, 143–170 (1984).
9. A. I. Madunts, "Classification of generalized formal Lubin–Tate groups over multidimensional local fields," *Zap. Nauchn. Semin. POMI*, **405**, 91–97 (2017).
10. A. I. Madunts, S. V. Vostokov, and R. P. Vostokova"Formal groups over subrings of the ring of integers of a multidimensional local field," *Vestn. SPbGU, Ser. Mat., Mekh., Astr.*, **6**, No. 1, 88–97 (2019).