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Perturbations of Dynamical Systems on Simple Time Scales

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Abstract—We study perturbations of dynamical systems in Banach spaces for which time varies on simple time scales consisting of families of isolated segments of the real axis. On a segment of the time scale, the system is governed by an ordinary differential equation; the transfer of a trajectory from a segment to the next one is determined by a map of the Banach space. The main problem which we study is the following one: given a trajectory of the original system, can we find a close trajectory of a perturbed system? We study perturbations applying the so-called multiscale approach: it is assumed that there exists a countable family of projections of the phase space and the smallness conditions are imposed on the projections of perturbations. To find a solution close to a specified solution of the unperturbed system, we introduce a generalization of the Perron method.

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1. INTRODUCTION

Dynamical systems on time scales are now a widely studied branch of the modern theory of dynamical systems. Such systems have been introduced by Aulbach and Hilger [1]. Surveys of the results obtained in the area can be found in [2–4], and [5]. Of course, it is as important to study various classes of perturbations for time scale systems as well as for ordinary differential equations or discrete dynamical systems.

In this paper, we study perturbations of dynamical systems in Banach spaces on so-called simple time scales that are families of isolated segments. On any of these segments, the system is governed by a differential equation, while the transition from a segment to the next one is determined by a map of the space. Such systems are naturally related to the so-called hybrid systems intensively studied in control theory (see, for example, the book [6]).

We study perturbations of such systems applying the so-called multiscale approach: it is assumed that there exists a countable family of projections of the phase space; the smallness conditions are imposed on the projections of perturbations (the idea develops the approach introduced by the author in the paper [7]). To find a solution close to a specified solution of the unperturbed system, we apply a generalization of the Perron method.

2. PERTURBATIONS OF DYNAMICAL SYSTEMS ON SIMPLE TIME SCALES: MULTISCALE APPROACH

Let us consider a simple variant of a time scale $\mathcal{T}$ that is a subset of $[0, \infty)$ and consists of isolated segments T_n, where $n = 1, 2, \dots, T_n = [l_n, r_n]$, and $0 \le l_1 < r_1 < l_2 < \dots$.

The phase space is a Banach space X with norm $|\cdot|$. We denote by $||\cdot||$ the operator norm of a linear operator.

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The system on T_n is generated by a differential equation

$$\dot{x} = A_n(t)x + a_n(t,x), \quad t \in T_n. \tag{1}$$

We assume that the operators $A_n(t)$ are continuous and bounded on T_n. The functions $a_n(t,x)$ are assumed to be continuous and Lipschitz continuous in x on $T_n \times X$ (with small Lipschitz constants).

Denote by $\phi_n(t,t_0,x_0)$ the solution of the Cauchy problem (t_0,x_0) for system (1).

Thus, for any $t_0 \in T_n$ and for any $x_0 \in X$ there exists a solution $\phi_n(t,t_0,x_0)$ defined on the whole segment T_n (in what follows, we work with such solutions).

Let $\Phi_n(t)$ and $\Psi_n(t)$ be the fundamental matrices of the system

$$\dot{x} = A_n(t)x$$

such that $\Phi_n(l_n) = E$ and $\Psi_n(r_n) = E$, respectively, where E is the identity map of X.

For any index $n = 1, 2, \dots$ we fix a map of X taking a point x to $B_n x + b_n(x)$, where B_n is a linear operator and $b_n(x)$ is a continuous function (it is not assumed, in general, that any operator B_n is an isomorphism of the space X).

The trajectory $x(t)$, $t \in \mathcal{T}$, of the appearing system starting at a point $x_0 \in X$ at the time moment l_1 (the left-hand end of the segment T_1) is defined as follows:

- $x(t) = \phi_1(t, l_1, x_0)$, $t \in T_1$,
- $x(l_2) = B_1 x(r_1) + b_1(x(r_1))$,
- $x(t) = \phi_2(t, l_2, x(l_2))$, $t \in T_2$,
- $x(l_3) = B_2 x(r_2) + b_2(x(r_2))$, and so on.

We fix countable sets of indices K^- and K^+ and assume that there exist families of continuous projections $P_k(t)$, $t \in \mathcal{T}$, of the space X indexed by $k \in K = K^- \bigcup K^+$ and having the following properties (2)–(6).

Let

$$P^-(t) = \sum_{k \in K^-} P_k(t) \quad \text{and} \quad P^-(t) = \sum_{k \in K^+} P_k(t).$$

We assume that

$$P^-(t) + P^+(t) = E, \qquad t \in \mathcal{T}; \tag{2}$$

$$P^-(t)P^+(t) = 0, \qquad t \in \mathcal{T}; \tag{3}$$

$$\Phi_n(t)P_k(l_n) = P_k(t)\Phi_n(t) \quad \text{and} \quad \Phi_n^{-1}(t)P_k(t) = P_k(l_n)\Phi_n^{-1}(t), \quad t \in T_n; \quad k \in K^+; \tag{4}$$

$$\Psi_n(t)P_k(r_n) = P_k(t)\Psi_n(t) \quad \text{and} \quad \Psi_n^{-1}(t)P_k(t) = P_k(r_n)\Psi_n^{-1}(t), \quad t \in T_n; \quad k \in K^-; \tag{5}$$

$$B_n P_k(r_n) = P_k(l_{n+1})B_n, \quad n \ge 1, \quad k \in K. \tag{6}$$

Concerning the projections P_k, $k \in K^-$, we assume, in addition, that the following property I holds: the restriction of any map B_n, $n \ge 1$, to the subspace $P_k(r_n)X$, $k \in K^-$, is an isomorphism of the subspace $P_k(r_n)X$ to the space $P_k(l_{n+1})X$.

For a trajectory $x(t)$, we denote $y_k(t) = P_k(t)x(t)$, $k \in K^+$, and $z_k(t) = P_k(t)x(t)$, $k \in K^-$.

Analogs of the Perron operator. First of all, we write down formulas for the analog of the "direct" Perron operator; our goal is to express for a trajectory $x(t)$, the value $y_k(t)$ for $t \in T_n$ in terms of Φ_m, B_m, $a_m(s, x(s))$ with $s \in T_m$, and $b_m(x(r_m))$, where $m \le n$ (thus, avoiding direct dependence on $x(s)$).

We assume that $x_0 = x(l_1) = 0$ (this will be enough for our purposes). Then,

$$x(t) = \int_{l_1}^{t} \Phi_1(t)\Phi_1^{-1}(s)a_1(s, x(s))\, ds, \quad t \in T_1,$$

and

$$x(r_1) = \int_{l_1}^{r_1} \Phi_1(r_1)\Phi_1^{-1}(s)a_1(s, x(s))\, ds.$$

Hence,

$$x(l_2) = B_1 x(r_1) + b_1(x(r_1)) = B_1 \int_{l_1}^{r_1} \Phi_1(r_1)\Phi_1^{-1}(s)a_1(s, x(s))\, ds + b_1(x(r_1)),$$

$$x(t) = \Phi_2(t)\left(B_1 \int_{l_1}^{r_1} \Phi_1(r_1)\Phi_1^{-1}(s)a_1(s, x(s))\, ds + \int_{l_2}^{t} \Phi_2^{-1}(s)a_2(s, x(s))\, ds + b_1(x(r_1))\right)$$

for $t \in T_2$, and

$$x(r_2) = \Phi_2(r_2)\left(B_1 \int_{l_1}^{r_1} \Phi_1(r_1)\Phi_1^{-1}(s)a_1(s, x(s))\, ds + \int_{l_2}^{r_2} \Phi_2^{-1}(s)a_2(s, x(s))\, ds + b_1(x(r_1))\right).$$

Continuing this process, we see that

$$x(t) = \Phi_3(t)\left(B_2\Phi_2(r_2)B_1 \int_{l_1}^{r_1} \Phi_1(r_1)\Phi_1^{-1}(s)a_1(s, x(s))\, ds + B_2 \int_{l_2}^{r_2} \Phi_2(r_2)\Phi_2^{-1}(s)a_2(s, x(s))\, ds\right.$$
$$\left. + \int_{l_3}^{t} \Phi_3^{-1}(s)a_3(s, x(s))\, ds + B_2\Phi_2(r_2)b_1(x(r_1)) + b_2(x(r_2))\right)$$

for $t \in T_3$, and so on. Thus, we get the following general formula

$$x(t) = \Phi_n(t)\left(B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2)B_1 \int_{l_1}^{r_1} \Phi_1(r_1)\Phi_1^{-1}(s)a_1(s, x(s))\, ds\right.$$
$$+ B_{n-1}\Phi_{n-1}(r_{n-1})\cdots \Phi_3(r_3)B_2 \int_{l_2}^{r_2} \Phi_2(r_2)\Phi_2^{-1}(s)a_2(s, x(s))\, ds + \cdots$$
$$\left. + \int_{l_n}^{t} \Phi_n^{-1}(s)a_n(s, x(s))\, ds + B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2))b_1(x(r_1)) + \cdots + b_{n-1}(x(r_{n-1}))\right) \quad (7)$$

for $t \in T_n$. Denote the right-hand side of formula (7) by

$$\mathcal{P}_n^+(t, x, a_1, \ldots, a_n, b_1, \ldots, b_{n-1}).$$

We indicate the dependence of this value on $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$ since in the future we will work with various a_k and b_k, while Φ_k and B_k will be fixed).

It follows from relations (4) and (6) that

$$y_k(t) = \mathcal{P}_n^+(t, x, P_k a_1, \ldots, P_k a_n, P_k(l_2)b_1, \ldots, P_k(l_n)b_{n-1}) \quad (8)$$

for $t \in T_n$ and $k \in K^+$ (in this notation, $P_k a_1$ means $P_k(s)a_1(s, x(s))$ for $s \in T_1$, $P_k(l_2)b_1$ means $P_k(l_2)b_1(x(r_1))$, and so on).

Indeed, let us look, for example, at the term in the formula for $y_k(t)$ corresponding to the first summand in formula (7): the first equality in (4) implies that

$$y_k(t) = P_k(t)\Phi_n(t)B_{n-1}\Phi_{n-1}(r_{n-1})\ldots a_1(s, x(s)) + \ldots$$

$$= \Phi_n(t)P_k(l_n)B_{n-1}\Phi_{n-1}(r_{n-1})\dots a_1(s,x(s)) + \dots;$$

formula (6) implies now that

$$y_k(t) = \Phi_n(t)B_{n-1}P_k(r_{n-1})\Phi_{n-1}(r_{n-1})\dots a_1(s,x(s)) + \dots,$$

and so on. Continuing similar reasoning, we prove relation (8).

Now we show how to find the “inverse” Perron operator expressing $z_k(t) = P_k x(t)$ for $t \in T_n$ and $k \in K^-$ in terms of Ψ_m, B_m, $a_m(s,x(s))$ with $s \in T_m$, and $b_m(x(r_m))$, where $m \geq n$.

Let us recall that, by assumption, property I holds: the restriction of any map B_n, $n \geq 1$, to the subspace $P_k(r_n)X$, $k \in K^-$, is an isomorphism of the subspace $P_k(r_n)X$ to the space $P_k(l_{n+1})X$.

We start with the formula

$$x(t) = \Psi_n(t)\left(x(r_n) - \int_t^{r_n} \Psi_n^{-1}(s)a_n(s,x(s))\,ds\right), \quad t \in T_n,$$

and, taking into account property (5), apply to this equality the projection $P_k(t)$ with $k \in K^-$ to show that

$$z_k(t) = \Psi_n(t)\left(z_k(r_n) - \int_t^{r_n} \Psi_n^{-1}(s)P_k(s)a_n(s,x(s))\,ds\right), \quad t \in T_n. \tag{9}$$

Applying $P_k(l_{n+1})$ to the equality

$$x(l_{n+1}) = B_n x(r_n) + b_n(x(r_n)),$$

and taking into account formula (6), we see that

$$z_k(l_{n+1}) = B_n z_k(r_n) + P_k(l_{n+1})b_n(x(r_n)),$$

and now property I implies that

$$z_k(r_n) = B_n^{-1}z_k(l_{n+1}) - B_n^{-1}P_k(l_{n+1})b_n(x(r_n)).$$

On T_{n+1},

$$z_k(t) = \Psi_{n+1}(t)\left(z_k(r_{n+1}) - \int_t^{r_{n+1}} \Psi_{n+1}^{-1}(s)P_k(s)a_{n+1}(s,x(s))\,ds\right);$$

hence,

$$z_k(l_{n+1}) = \Psi_{n+1}(l_{n+1})\left(z_k(r_{n+1}) - \int_{l_{n+1}}^{r_{n+1}} \Psi_{n+1}^{-1}(s)P_k(s)a_{n+1}(s,x(s))\,ds\right). \tag{10}$$

Combining formulas (9) and (10), we get the following representation of $z_k(t)$ on T_n (with the explicit dependence on $z_k(r_n)$ in formula (9) replaced by the dependence on $z_k(r_{n+1})$):

$$z_k(t) = \Psi_n(t)\left(B_n^{-1}\left(\Psi_{n+1}(l_{n+1})\left(z_k(r_{n+1}) + \int_{r_{n+1}}^{l_{n+1}} \Psi_{n+1}^{-1}(s)P_k(s)a_{n+1}(s,x(s))\,ds\right)\right.\right.$$
$$\left.\left. - P_k(l_{n+1})b_n(x(r_n))\right) + \int_{r_n}^{t} \Psi_n^{-1}(s)P_k(s)a_n(s,x(s))\,ds\right). \tag{11}$$

Now we can apply an analog of formula (9) connecting $z_k(r_{n+1})$ and $z_k(l_{n+2})$ and an analog of formula (10) expressing $z_k(l_{n+2})$ in terms of Ψ_{n+2} and $z_k(r_{n+2})$ and to get a representation of $z_k(t)$ for $t \in T_n$ similar to (11).

This representation will include $\Psi_n, \Psi_{n+1}, \Psi_{n+2}, B_n, B_{n+1}, a_n, a_{n+1}, a_{n+2}, b_n, b_{n+1}$ (and $z_k(r_{n+2})$ instead of $z_k(r_{n+1})$).

Let us write down the corresponding terms (we exclude the term containing $z_k(r_{n+2})$ since this term will disappear in the future induction process while the terms below will be preserved):

• including a_n:

$$\Psi_n(t)\int\limits_{r_n}^{t}\Psi_n^{-1}(s)P_k(s)a_n(s,x(s))\,ds;$$

• including a_{n+1}:

$$\Psi_n(t)B_n^{-1}\Psi_{n+1}(l_{n+1})\int\limits_{r_{n+1}}^{l_{n+1}}\Psi_{n+1}^{-1}(s)P_k(s)a_{n+1}(s,x(s))\,ds;$$

• including a_{n+2}:

$$\Psi_n(t)B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}\Psi_{n+2}(l_{n+2})\int\limits_{r_{n+2}}^{l_{n+2}}\Psi_{n+2}^{-1}(s)P_k(s)a_{n+2}(s,x(s))\,ds;$$

• including b_n:

$$-\Psi_n(t)B_n^{-1}P_k(l_{n+1})b_n(x(r_n));$$

• including b_{n+1}:

$$-\Psi_n(t)B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}P_k(l_{n+2})b_{n+1}(x(r_{n+1})).$$

Continuing this (eventually infinite) process, we get a formula

$$z_k(t) = \mathcal{P}_n^-(t, x, P_k a_n, P_k a_{n+1}, \dots, P_k(l_{n+1})b_n, P_k(l_{n+2})b_{n+1}, \dots), \quad t \in T_n, \tag{12}$$

similar to (8).

The right-hand side of formula (12) is an infinite series. This series obviously converges for any trajectory $x(t)$ of our system (just because $P_k(t)x(t)$ is defined for all $t \in \mathcal{T}$).

Perturbations. The first natural statement of the perturbation problem is as follows.

We replace systems (1) on the segments T_n by systems

$$\dot{x} = C_n(t)x + c_n(t,x)$$

(assuming that the operators $C_n(t)$ and the functions $c_n(t,x)$ have properties similar to those of $A_n(t)$ and $a_n(t,x)$) and the maps $B_n x + b_n(x)$ by similar maps $D_n x + d_n(x)$, take a trajectory $\xi(t)$ of the new system and look for a close trajectory $x(t)$ of the original system.

As usual, we are looking for functions $v_n(t)$ on T_n with values in X such that

$$x(t) = \xi(t) + v(t), \quad t \in T_n.$$

From the relations

$$\dot{x} = A_n(t)x + a_n(t,x) = A_n(t)(\xi + v) + a_n(t, \xi + v) = \dot{\xi} + \dot{v} = C_n(t)\xi + c_n(t,\xi) + \dot{v}$$

we deduce the equations for v_n:

$$\dot{v} = A_n(t)v + a_n^*(t,\xi,v), \quad t \in T_n, \tag{13}$$

where

$$a_n^*(t,\xi,v) = A_n(t)\xi + a_n(t,\xi+v) - C_n(t)\xi - c_n(t,\xi),$$

which we represent in the form

$$a_n^*(t,\xi,v) = A_n(t)\xi + a_n(t,\xi+v) - a_n(t,\xi) - C_n(t)\xi + a_n(t,\xi) - c_n(t,\xi),$$

and the summands of the right-hand side of the above formula have the following properties: $a_n(t,\xi+v)-a_n(t,\xi)$ vanishes for $v=0$ and has small Lipschitz constant in v for small $|v|$ (of course, if we impose a similar condition on $a_k(t,x)$) and $a_n(t,\xi)-c_n(t,\xi)$ is small (if the perturbed system is close to the nonperturbed one).

Now we look at the "transition rule." From the equalities

$$\begin{aligned} x(l_{n+1}) &= B_n x(r_n) + b_n(x(r_n)) = B_n\xi(r_n) + B_n v(r_n) + b_n(\xi(r_n)+v(r_n)) \\ &= \xi(l_{n+1}) + v(l_{n+1}) = D_n\xi(r_n) + d_n(\xi(r_n)) + v(l_{n+1}) \end{aligned}$$

we deduce the relations

$$v(l_{n+1}) = B_n v(r_n) + b_n^*(\xi(r_n), v(r_n)), \quad n\ge 1, \tag{14}$$

where

$$b_n^*(\xi(r_n),v) = (B_n-D_n)\xi(r_n) + b_n(\xi(r_n)+v(r_n)) - b_n(\xi(r_n)) + b_n(\xi(r_n)) - d_n(\xi(r_n)).$$

Thus, for $v(t)$ we get system (13)–(14) similar to the original one (with the same $A_n(t)$ and B_n but, of course, with different "small" nonlinear terms).

We solve this system in a standard way.

Let $\mathcal{V}$ be the space of continuous functions on $\mathbb{T}$ with values in X and with the norm

$$||v|| = \sup_{n\ge 1}\max_{t\in T_n}|v(t)|.$$

Clearly, $\mathcal{V}$ is a complete metric space with the metric $\rho(v,w)=||v-w||$.

Our goal is to indicate conditions under which the "Perron operator" corresponding to system (13) and (14) has a fixed point in V whose norm we can control.

Main assumption. We make the following main assumption.

There exist sequences of positive numbers $\alpha_{n,k},\beta_{n,k}$ and a number $M>0$ such that

$$\begin{aligned} ||\Phi_n(t)|| \Bigg(\alpha_{1,k} \left|\left| B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2)B_1\int_{l_1}^{r_1}\Phi_1(r_1)\Phi_1^{-1}(s)P_k(s)\,ds\right|\right| \\ + \alpha_{2,k}\left|\left| B_{n-1}\Phi_{n-1}(r_{n-1})\cdots\Phi_3(r_3)B_2\int_{l_2}^{r_2}\Phi_2(r_2)\Phi_2^{-1}(s)P_k(s)\,ds\right|\right| + \cdots \\ + \alpha_{n,k}\left|\left|\int_{l_n}^{t}\Phi_n^{-1}(s)P_k\,ds\right|\right| + \beta_{1,k}||B_{n-1}\Phi_{n-1}(r_{n-1})\cdots B_2\Phi_2(r_2))P_k(l_2)|| + \cdots + \beta_{n,k}||P_k(l_{n+1})||\Bigg) \\ \le M, \quad t\in T_n, \quad n\ge 1, \quad k\in K^+, \end{aligned} \tag{15}$$

and

$$\begin{aligned} ||\Psi_n(t)|| \Bigg(\alpha_{n,k}\left|\left|\int_{r_n}^{t}\Psi_n^{-1}(s)P_k\,ds\right|\right| + \alpha_{n+1,k}||B_n^{-1}\Psi_{n+1}(l_{n+1})\int_{r_{n+1}}^{l_{n+1}}\Psi_{n+1}^{-1}(s)P_k(s)\,ds|| \\ + \alpha_{n+2,k}\left|\left|B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}\Psi_{n+2}(l_{n+2})\int_{r_{n+2}}^{l_{n+2}}\Psi_{n+2}^{-1}(s)P_k(s)\,ds\right|\right| + \dots \\ + \beta_{n,k}||B_n^{-1}P_k(l_{n+1})|| + \beta_{n+1,k}||B_n^{-1}\Psi_{n+1}(l_{n+1})B_{n+1}^{-1}P_k(l_{n+2})|| + \dots \Bigg) \le M, \\ t\in T_n, \quad n\ge 1, \quad k\in K^-. \end{aligned} \tag{16}$$

Condition on perturbations. We impose the following smallness conditions on the perturbations: there exists a $d_0 > 0$ such that if $d \le d_0$ and $|v| \le 2Md$, then

$$|P_k(t)a_n^*(t,\xi(t),v)| \le \alpha_{n,k}d, \tag{17}$$

$$|P_k(l_{n+1})b_n^*(\xi(r_n),v)| \le \beta_{n,k}d, \tag{18}$$

$$\mathrm{Lip}_v\,(P_k(t)a_n^*(t,\xi(t),v)) \le \frac{\alpha_{n,k}}{4M}, \tag{19}$$

and

$$\mathrm{Lip}_v\,(P_k(l_{n+1})b_n^*(\xi(r_n),v)) \le \frac{\beta_{n,k}}{4M}. \tag{20}$$

for $t \in T_n$, $n \ge 1$, and $k \in K$, where Lip_v is a Lipschitz constant in variable v.

Consider the following operator Θ on the space $\mathcal{V}$: Θ maps a $V \in \mathcal{V}$ to a function $W \in \mathcal{V}$ determined by its projections

$$(P_kW)(t) = \mathcal{P}_n^+(t, v, P_ka_1^*, \dots, P_k(l_2)b_1^*, \dots), \quad t \in T_n, k \in K^+,$$

and

$$(P_kW)(t) = \mathcal{P}_n^-(t, v, P_ka_n^*, \dots, P_k(l_{n+1})b_n^*, \dots), \quad t \in T_n, k \in K^-,$$

where the $\mathcal{P}_n^+$ and $\mathcal{P}_n^-$ are defined in (8) and (12), respectively, $a_n^* = a_n^*(t,\xi,v(t))$, and so on.

Let us estimate, for example, one of the terms in the expression for $(P_kW)(t)$. Take

$$V \in \mathcal{V}_{2Md} := \{V :\ ||V|| \le 2Md\}.$$

Since $P_k(t)$ is a projection, we deduce from (17) that

$$\left|\int_{l_n}^{t} \Phi_n^{-1}(s)P_k(s)a_n^*(t,\xi(t),v)\ ds\right| = \left|\int_{l_n}^{t} \Phi_n^{-1}(s)P_k(s)P_k(s)a_n^*(t,\xi(t),v)\ ds\right|$$
$$\le \alpha_{n,k}d\left|\left|\int_{l_n}^{t} \Phi_n^{-1}(s)P_k(s)\,ds\right|\right|.$$

Now estimates (15)–(18) imply that if $V \in \mathcal{V}_{2Md}$, then

$$\left|\left|\sum_{k\in K^+} P_kW\right|\right|, \left|\left|\sum_{k\in K^-} P_kW\right|\right| \le Md;$$

hence,

$$||W|| = \left|\left|\sum_{k\in K} P_kW\right|\right| \le 2Md,$$

and Θ maps $\mathcal{V}_{2Md}$ into itself.

It follows from (19) and (20) that if $V, V' \in \mathcal{V}_{2Md}$, $W = \Theta(V)$, and $W' = \Theta(V')$, then $||W - W'|| \le \frac{1}{2}||V - V'||$, i.e., Θ is a contraction on $\mathcal{V}_{2Md}$.

The fixed point V of Θ in $\mathcal{V}_{2Md}$ gives us a trajectory $x(t) = \xi(t) + v(t)$ of the unperturbed system such that

$$|x(t) - \xi(t)| \le 2Md, \quad t \in \mathbb{T}. \tag{21}$$

Thus, we have proved the following result.

Theorem 1. *Under our main assumptions (15) and (16), the smallness conditions (17)–(20) imply that for the trajectory $\xi(t)$ of the perturbed system there exists a trajectory $x(t)$ of the unperturbed system satisfying estimate (21).*

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