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On the Existential Arithmetics with Addition and Bitwise Minimum

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Counters

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- For $k \geq 2$ consider FA $\mathcal{A}$ over Σ_k^n for $\Sigma_k = \{0, 1, \dots, k-1\}$.
- The language $L(\mathcal{A})$ and the set $[\![L(\mathcal{A})]\!]_k \subseteq \mathbb{N}^n$.
- $R \subseteq \mathbb{N}^n$ is called <u>k-FA-recognizable</u> if there exists Σ_k^n-FA $\mathcal{A}$ such that $R = [\![L(\mathcal{A})]\!]_k$.

Theorem. Büchi [1960], Bruyère [1985], Villemaire [1992]: $R \subseteq \mathbb{N}^n$ is k-FA-recognizable if and only if it is $\exists\forall\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, V_k, = \rangle$, where $V_k(x, y)$ iff x is the largest power of k that divides y.

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Question: Whether there is a "natural" structure where every k-FA-recognizable relation is $\exists$-definable, and vice versa?

- Denote by $\mathcal{F}$ the set of all finite subsets of $\mathbb{N}$.
- 2-FA-recognizability of $R \subseteq \mathcal{F}^n$ is defined similarly.

Theorem. Büchi [1960], Elgot [1961], Trakhtenbrot [1962]: $R \subseteq \mathcal{F}^n$ is 2-FA-recognizable iff it is WMSO-definable in the structure $\langle \mathbb{N}; S \rangle$.
Therefore, WMSOTh$\langle \mathbb{N}; S \rangle$ is decidable.

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- Decidability of the Emptiness problem and undecidability of the Universality problem for Parikh automata.

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Question: Is there a "reasonable" existential FO-characterization of Parikh automata?

- In these logical characterizations the universal quantifier is bounded.
- Davis, Putnam, and Robinson [1963]: Every relation $R \subseteq \mathbb{N}^n$ is r.e. if and only if it is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, \cdot, exp, = \rangle$.

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- Matiyasevich's proof of DPR-theorem [1976]: Purely existential arithmetization of Turing machines. The structure $\langle \mathbb{N}; 0, 1, +, \&, \frown, = \rangle$, for the bitwise minimum operation & and concatenation $\frown$, where $t = x \frown y \rightleftharpoons t = x + 2^{l(x)} y$.

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$$y = \Theta_{k,a}(x) \Leftrightarrow \exists x_1 ... \exists x_{k-1} \Big(\bigwedge_{1 \le i < j \le k-1} x_i \&_k x_j = 0 \ \wedge (x_1 + ... + x_{k-1}) \preccurlyeq_k \mathbf{1}_k(x) \wedge$$
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$y = \Theta_{k,0}(t, x)$. **Example:** $\Theta_{3,0}(100000, 1020) = 110101$

Existential characterization of k-FA-recognizable languages

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- k-FA $\mathcal{A} = (Q, q_0, F, \delta)$.
- Variables $\overline{q} = q_0, ..., q_s$ for every $q_i \in Q$.
- For a state $p \in Q$, denote by $\nu(p)$ its number from $[0..s]$.

$$K_k(t, \overline{q}) \rightleftharpoons \bigwedge_{0 \leq i < j \leq s} q_i \&_k q_j = 0 \wedge q_0 + ... + q_s = \mathbf{1}_k(t) \wedge 1 \preccurlyeq_k q_0 \wedge \bigvee_{p \in F} t \preccurlyeq_k q_{\nu(p)}.$$

- For every $(p, \overline{a}) \in Q \times \Sigma_k^n$

$$\Delta_{(p,\overline{a})}(t, \overline{q}, \overline{x}) \rightleftharpoons \left(q_{\nu(p)} \&_k \mathop{\&_k}_{i \in [\mathbf{1}..n]} \Theta_{k,a_i}(\frac{t}{k}, x_i) \right) \preccurlyeq_k \left(\mathop{\Big|_k}_{\widetilde{p} \in \delta(p, \overline{a})} \frac{q_{\nu(\widetilde{p})}}{k} \right).$$

$$R_{L(\mathcal{A})}(\overline{x}) \Leftrightarrow \exists t \exists \overline{q} \Big(P_k(t) \wedge \bigwedge_{i \in [\mathbf{1}..n]} x_i < t \wedge K_k(t, \overline{q})$$

Existential characterization of k-FA-recognizable languages

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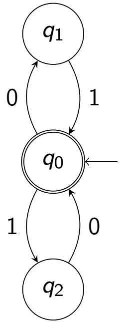
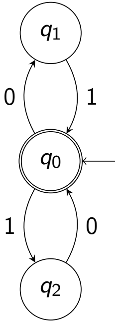
$$R_{L(\mathcal{A})}(\overline{x}) \Leftrightarrow \exists t \exists \overline{q} \Big(P_k(t) \wedge \bigwedge_{i \in [1..n]} x_i < t \wedge K_k(t, \overline{q}) \wedge \bigwedge_{(p,\overline{a}) \in Q \times \Sigma_k^n} \Delta_{(p,\overline{a})}(t, \overline{q}, \overline{x}) \Big).$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$



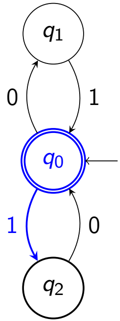
Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	1	0	0	1	0	1	x
···	0										t
···	0										q_0
···	0										q_1
···	0										q_2



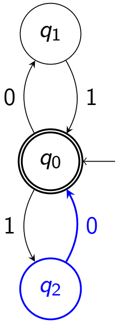
Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

⋯	0	0	0	1	1	0	0	1	0	1	x
⋯	0										t
⋯	0									1	q_0
⋯	0										q_1
⋯	0								1		q_2



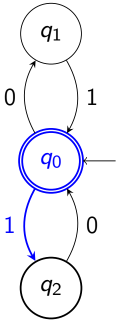
Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

$\cdots$	0	0	0	1	1	0	0	1	**0**	1	x
$\cdots$	0										t
$\cdots$	0							**1**		1	q_0
$\cdots$	0										q_1
$\cdots$	0								**1**		q_2



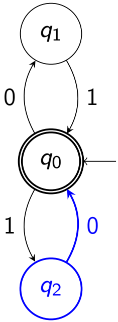
Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	1	0	0	**1**	0	1	x
···	0										t
···	0							**1**		1	q_0
···	0										q_1
···	0						**1**		1		q_2

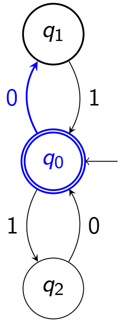


Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

$\cdots$	0	0	0	1	1	0	**0**	1	0	1	x
$\cdots$	0										t
$\cdots$	0					**1**		1		1	q_0
$\cdots$	0										q_1
$\cdots$	0						**1**		1		q_2

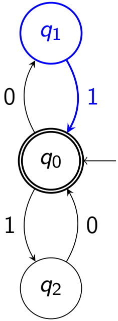


···	0	0	0	1	1	**0**	0	1	0	1	x
···	0										t
···	0					**1**		1		1	q_0
···	0				**1**						q_1
···	0						1		1		q_2



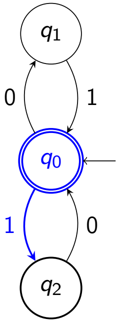
Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	**1**	0	0	1	0	1	x
···	0										t
···	0			**1**		1		1		1	q_0
···	0				**1**						q_1
···	0						1		1		q_2



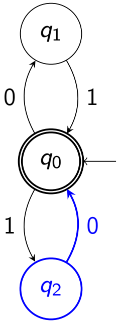
Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

$\cdots$	0	0	0	**1**	1	0	0	1	0	1	x
$\cdots$	0										t
$\cdots$	0			**1**		1		1		1	q_0
$\cdots$	0				1						q_1
$\cdots$	0		**1**				1		1		q_2



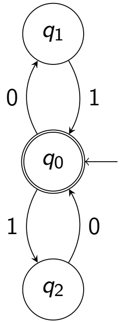
Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

⋯	0	0	**0**	1	1	0	0	1	0	1	x
⋯	0	1									t
⋯	0	**1**		1		1		1		1	q_0
⋯	0				1						q_1
⋯	0		1				1		1		q_2



Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

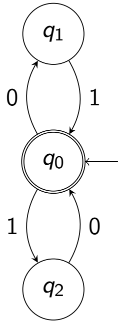
···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$

Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

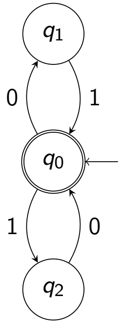
···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



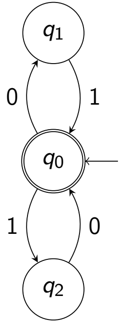
$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$

$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$

$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

$\cdots$	0	0	0	**1**	1	0	0	**1**	0	**1**	x
$\cdots$	0	1									t
$\cdots$	0	1		**1**		1		**1**		**1**	q_0
$\cdots$	0				1						q_1
$\cdots$	0			**1**				**1**		**1**	$\frac{q_2}{2}$



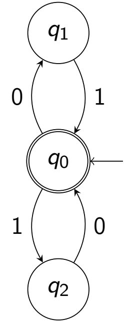
$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$

$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$

$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \wedge$$

Example: $\exists$-formula for the set $[\![\{10, 01\}^*]\!]_2$

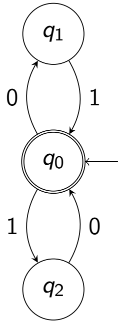
$\cdots$	0	0	0	1	1	0	0	1	0	1	x
$\cdots$	0	1									t
$\cdots$	0	1		1		1		1		1	q_0
$\cdots$	0				1						q_1
$\cdots$	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$
$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \ \wedge$$

Example: ∃-formula for the set $[\![\{10, 01\}^*]\!]_2$

···	0	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**0**	$\sim(\frac{t}{2}, x)$
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0					1					$\frac{q_1}{2}$
···	0		1				1		1		q_2



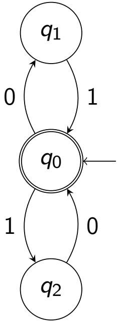
$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \land x < t \land q_0 + q_1 + q_2 = 2t - 1 \land$$

$$q_0 \& q_1 = 0 \land q_0 \& q_2 = 0 \land q_1 \& q_2 = 0 \land$$

$$q_0 \& 1 = 1 \land q_0 \& t = t \land q_0 \& x \preccurlyeq \frac{q_2}{2} \land q_0 \& \sim(\frac{t}{2}, x) \preccurlyeq \frac{q_1}{2} \land$$

Example: $\exists$-formula for the set $[\![\{10,01\}^*]\!]_2$

⋯	0	0	0	1	1	0	0	1	0	1	x
⋯	0	1									t
⋯	0	1		1		1		1		1	q_0
⋯	0				1						q_1
⋯	0		1				1		1		q_2



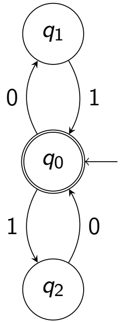
$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$

$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$

$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \wedge q_0 \& \sim (\frac{t}{2}, x) \preccurlyeq \frac{q_1}{2} \wedge$$

$$q_1 \& x \preccurlyeq \frac{q_0}{2} \wedge q_1 \& \sim (\frac{t}{2}, x) \preccurlyeq 0 \wedge q_2 \& x \preccurlyeq 0 \ \wedge$$

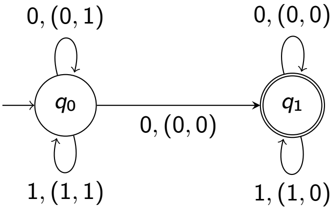
···	0	0	0	1	1	0	0	1	0	1	x
···	0	1									t
···	0	1		1		1		1		1	q_0
···	0				1						q_1
···	0		1				1		1		q_2



$$\exists t \exists q_0 \exists q_1 \exists q_2 \Big(P_2(t) \wedge x < t \wedge q_0 + q_1 + q_2 = 2t - 1 \wedge$$
$$q_0 \& q_1 = 0 \wedge q_0 \& q_2 = 0 \wedge q_1 \& q_2 = 0 \wedge$$
$$q_0 \& 1 = 1 \wedge q_0 \& t = t \wedge q_0 \& x \preccurlyeq \frac{q_2}{2} \wedge q_0 \& \sim (\frac{t}{2}, x) \preccurlyeq \frac{q_1}{2} \wedge$$
$$q_1 \& x \preccurlyeq \frac{q_0}{2} \wedge q_1 \& \sim (\frac{t}{2}, x) \preccurlyeq 0 \wedge q_2 \& x \preccurlyeq 0 \wedge q_2 \& \sim (\frac{t}{2}, x) \preccurlyeq \frac{q_0}{2} \Big).$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

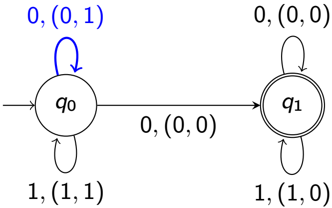
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 0$
$y_2 = 0$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

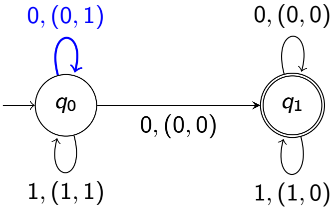
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$
\begin{aligned}
x &= 10011011100 \\
y_1 &= 0 \\
y_2 &= 1
\end{aligned}
$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

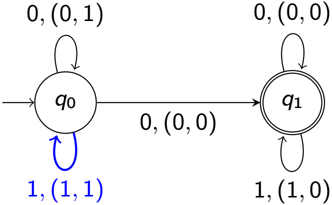
$\{0,1\}$-PFA with
$D = \{(0,0),(0,1),(1,0),(1,1)\}$ and
$\varphi \rightleftharpoons x = y$.



$$
\begin{aligned}
x &= 10011011100 \\
y_1 &= 0 \\
y_2 &= 2
\end{aligned}
$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

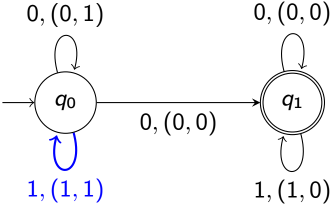
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 1$
$y_2 = 3$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.

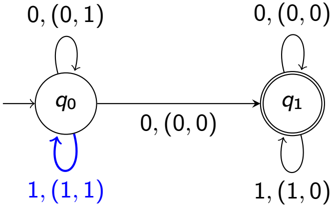


$x = 10011011100$
$y_1 = 2$
$y_2 = 4$

Parikh automata and $EqNZB_k$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

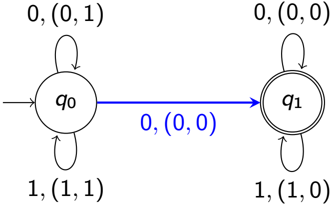
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 3$
$y_2 = 5$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

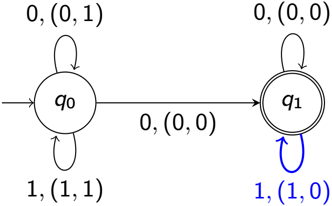
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$x = 10011011100$$
$$y_1 = 3$$
$$y_2 = 6$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

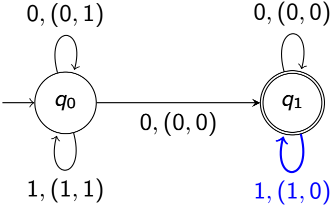
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$
\begin{aligned}
x &= 10011011100 \\
y_1 &= 4 \\
y_2 &= 6
\end{aligned}
$$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

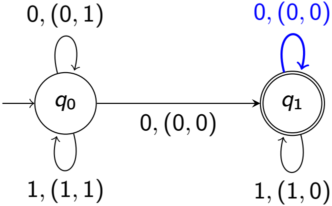
$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$x = 10011011100$
$y_1 = 5$
$y_2 = 6$

$L = \{x \in \{0,1\}^* \mid x_{\#_1(x)} = 0\}$, where x_i is the i-th letter of x.

$\{0,1\}$-PFA with $D = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\varphi \rightleftharpoons x = y$.



$$x = 10011011100$$
$$y_1 = 5$$
$$y_2 = 6$$

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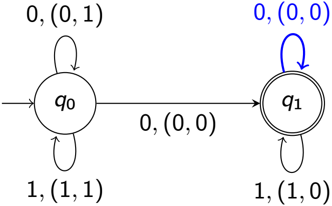

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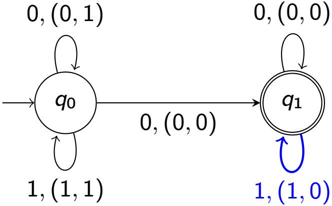

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$y_1 = 6$
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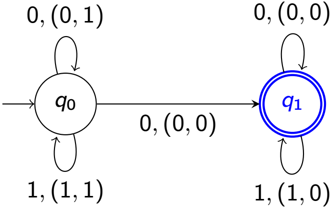

x = **10011011100**
$y_1 = 6$
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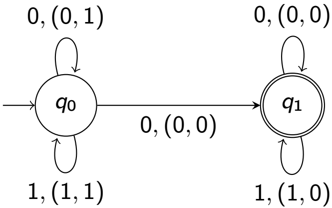

- Parikh map $\Phi_k : \mathbb{N} \to \mathbb{N}^k$ such that $\Phi_k(x) = (\#_{k,0}(x), \dots, \#_{k,k-1}(x))$, where $\#_{k,i}$ counts the number of occurrences of i in k-ary expansion of x.
- $R(x_1, \dots, x_n)$ is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, = \rangle$, and $\bar{a} \in \{0, \dots, k-1\}^n$. Then $R(\#_{k,a_1}(x_1), \dots, \#_{k,a_n}(x_n))$ is $\exists$-definable in $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, = \rangle$.

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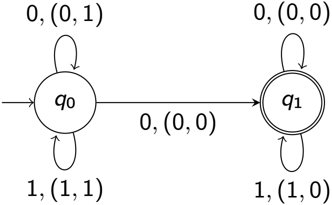

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$$\#_{k,a}(x) + \#_{k,b}(y) = \#_{k,c}(z) \Leftrightarrow \exists x' \exists y' (EqNZB_k(x' + y', \Theta_{k,c}(z)) \wedge x' \&_k y' = 0 \wedge EqNZB_k(x', \Theta_{k,a}(x)) \wedge EqNZB_k(y', \Theta_{k,b}(y))).$$

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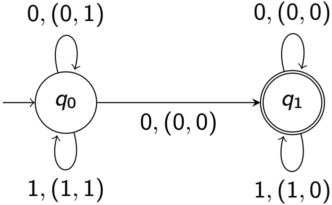

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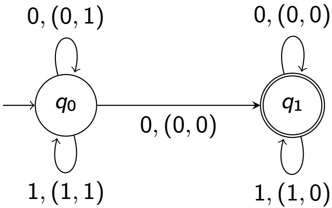

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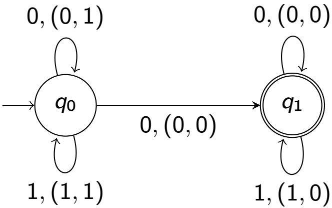

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- D is a **finite** subset of $\mathbb{N}^m$.
- $M(D)$ is the maximal element of D.
- Introduce $m(M(D)+1)$ variables $\overline{y} = y_{1,0},...,y_{1,M(D)},...,y_{m,0},...,y_{m,M(D)}$ such that for every $i \in [1..m]$ it holds that $\theta_k(t, y_{i,0}, ..., y_{i,M(D)})$, where

$$\theta_k(t, y_0, ..., y_M) \rightleftharpoons \bigwedge_{0 \le i < j \le M} y_i \&_k y_j = 0 \wedge y_0 + ... + y_M = \mathbf{1}_k(t).$$

Theorem 2

For every integer $k \geq 2$ a relation $R \subseteq \mathbb{N}^n$ is k-PFA-recognizable if and only if it is ∃-definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, = \rangle$.

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Corollary 1. The $\exists$-theory of $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, = \rangle$ is decidable and the $\forall\exists$-theory of this structure is undecidable. [Klaedtke and Rueß, 2003]

Theorem 2

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Corollary 2. The problem of deciding whether a set existentially definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, EqNZB_k, = \rangle$ is ∃-definable in $\langle \mathbb{N}; 0, 1, +, \&_k, = \rangle$ is undecidable. [Cadilhac, Finkel and McKenzie, 2011]

Multi-counter machines, concatenation, and DPR-theorem

Two-way multi-counter machine $\mathcal{M}$ over Σ_k^n (k-MCM): (m, Q, q_0, F, δ)

- The number of the counters $m \geq 0$.
- Transition function δ from $Q \times (\Sigma \cup \{\vdash, \dashv\}) \times \{0,1\}^m$ to $2^{Q \times \{-1,0,1\}^{m+1}}$.
- Configuration on an input $\vdash x \dashv$ is a tuple $(q, \vdash x \dashv, i, y_1, ..., y_m)$.
- $(q, \vdash x \dashv, i, y_1, ..., y_m) \to (q', \vdash x \dashv, i + \Delta, y_1 + d_1, ..., y_m + d_m)$ if and only if $(q', \Delta, d_1, ..., d_m) \in \delta(q, a, [y_1 > 0], ..., [y_m > 0])$.
- Input $x \in \Sigma^*$ is accepted by $\mathcal{M}$ if for $\vdash x \dashv$ there is a computation $(q_0, \vdash x \dashv, 0, 0, ..., 0) \to ... \to (q_f, \vdash x \dashv, 0, 0, ..., 0)$ for $q_f \in F$

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- Input $x \in \Sigma^*$ is accepted by $\mathcal{M}$ if for $\vdash x \dashv$ there is a computation $(q_0, \vdash x \dashv, 0, 0, ..., 0) \rightarrow ... \rightarrow (q_f, \vdash x \dashv, 0, 0, ..., 0)$ for $q_f \in F \rightsquigarrow L(\mathcal{M})$.
- $R \subseteq \mathbb{N}^n$ is *k-MCM-recognizable* if there exists a k-MCM $\mathcal{M}$ such that $\forall \bar{a} \in \mathbb{N}^n \left(R(\bar{a}) \Leftrightarrow R_{L(\mathcal{M})}(\bar{a})\right)$

Two-way multi-counter machine $\mathcal{M}$ over Σ_k^n (k-MCM): (m, Q, q_0, F, δ)

- The number of the counters $m \geq 0$.
- Transition function δ from $Q \times (\Sigma \cup \{\vdash, \dashv\}) \times \{0,1\}^m$ to $2^{Q \times \{-1,0,1\}^{m+1}}$.
- Configuration on an input $\vdash x \dashv$ is a tuple $(q, \vdash x \dashv, i, y_1, \ldots, y_m)$.
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Aim: The same arguments as in the cases of k-FA and k-PFA for existential characterization of r.e. sets.

Multi-counter machines, concatenation, and DPR-theorem

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Aim: The same arguments as in the cases of k-FA and k-PFA for existential characterization of r.e. sets. Introduce concatenation $t = x \frown_k y \rightleftharpoons t = x + k^{l_k(x)} y$ and use **byte**wise multiplication instead of **bit**wise to encode δ.

- Function $Copy_k(u, t, x)$.

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- Predicate $\Delta_k(u, t, x)$, which is true when u is a power of k greater than k^2, x has the same u-byte-length as t and has the form
  $x = \underline{0000001}\ 0000010\ 0000100\ 0000010\ 0000100\ 0000010\ \underline{0000001}$

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$x = 0000001 \ldots 0..0001000..0\ 0..abcdefg..0 \ldots \quad \ldots \quad 0000001$

$\frac{(kx)}{u} = 0000000 \ldots \quad \ldots \quad 0..0010000..0$

$\frac{(x)}{u} = 0000000 \ldots \quad \ldots \quad 0..0001000..0$

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Sequence of configurations $\rightsquigarrow$ sequence of transitions via **byte**wise multiplication.

- Function $U_k(u, x)$. $U_2(100, 10\,000\,011\,000\,010) = 1\,000\,001\,000\,001$.

- k-MCM $\mathcal{M} = (m, Q, q_0, F, \delta)$

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$$K_k(u, t, \overline{q}) \rightleftharpoons \bigwedge_{0 \leq i < j \leq s} q_i \&_k q_j = 0 \wedge q_0 + ... + q_s = Copy_k(u, t, 1) \wedge$$

$$1 \preccurlyeq_k q_0 \wedge \bigvee_{p \in F} \Lambda_k(u, t) \preccurlyeq_k q_{\nu(p)}.$$

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$$C_{\mathcal{M}}(u, t, \overline{q}, \overline{x}, \overline{\theta}, h, \overline{y}) \rightleftharpoons P_k(u) \wedge \bigwedge_{i \in [1..n]} k^4 x_i \leq u \wedge u \leq t \wedge K_k(u, t, \overline{q}) \wedge$$

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$$K_k(u,t,\overline{q}) \rightleftharpoons \bigwedge_{0 \le i < j \le s} q_i \&_k q_j = 0 \wedge q_0 + \ldots + q_s = Copy_k(u,t,1) \wedge$$
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$$\theta_{\vdash} = Copy_k(u,t,1) \wedge \bigwedge_{i \in [1..n]} \Big(\theta_{i,0} = Copy_k(u,t,k\Theta_{k,0}(x_i + b_k(\overline{x})) \wedge$$
$$\bigwedge_{a \in [1..k-1]} \theta_{i,a} = Copy_k(u,t,k\Theta_{k,a}(x_i)) \Big) \wedge \theta_{\dashv} = Copy_k(u,t,kb_k(\overline{x})) \wedge$$

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$$\Delta_k(u, t, h) \wedge \bigwedge_{i \in [1..m]} \Delta_k(u, t, y_i).$$

- Assume $C_{\mathcal{M}}(u, t, \overline{q}, \overline{x}, \overline{\theta}, h, \overline{y})$
- Letter $(a_1, ..., a_n) \in \Sigma_k^n \cup \{\vdash, \dashv\}$, a state $p \in Q$, and a tuple $\overline{c} \in \{0, 1\}^m$
- Counters from $Y_{\overline{c}} = \{i \in [1..m] \mid c_i = 0\}$ are equal to zero and from $[1..m] \setminus Y_{\overline{c}}$ are non-zero

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Example: $\vdash 101 \dashv$

$$\theta_{1,1} = 0001010\ 0001010\ 0001010\ 0001010\ 0001010$$

$$h = 0000001\ 0000010\ 0000100\ 0000010\ 0000001$$

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Example: $\vdash$ 101 $\dashv$

$$\theta_{1,1} = 0001010\ 0001010\ 0001010\ 0001010\ 0001010$$

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Example: $\vdash$ 101 $\dashv$

$\theta_{1,1} =$ 0001010 0001010 0001010 0001010 0001010

$h =$ 000000**1** 0000010 0000100 0000010 0000001

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Example: $\vdash 101 \dashv$

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$$h = 0000001\ 0000010\ 0000100\ 0000010\ 0000001$$
$$\theta_{1,1} \&_2 h = 0000000\ 0000010\ 0000000\ 0000010\ 0000000$$
$$U_2(u, (\theta_{1,1} \&_2 h)) = 0000000\ 0000001\ 0000000\ 0000001\ 0000000$$

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Example: $h = 0000001\ 0000010\ 0000100\ 0000010\ 0000001$

$\frac{(k^{\mathbf{1}} h)}{u} = 0000000\ 0000010\ 0000100\ 0001000\ 0000100$

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Theorem 3

For every integer $k \geq 2$ a relation is k-MCM-recognizable if and only if it is $\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, \&_k, \frown_k, = \rangle$. Therefore, every relation $R \subseteq \mathbb{N}^n$ is r.e. iff it is $\exists$-definable in this structure.

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Corollary 2. Every relation $R \subseteq \mathbb{N}^n$ is r.e. if and only if it is $\exists$-definable in the structure $\langle \mathbb{N}; 0, 1, +, EqNZB, \frown, = \rangle$.

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- Easy: $\exists\mathrm{Def}\langle \mathbb{N}; 0, 1, +, \&_2, = \rangle = \exists\mathrm{Def}\langle \mathbb{N}; 0, 1, +, [\![\{10, 01\}^*]\!]_2, = \rangle$. How can we **describe** k-recognizable relations $\mathcal{R}_k \subseteq \mathbb{N}^n$ such that every k-FA-recognizable is ∃-definable in $\langle \mathbb{N}; 0, 1, +, \mathcal{R}_k, = \rangle$?

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Thank you for your attention !