Skew Howe duality and q-Krawtchouk polynomial ensemble

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Abstract

We consider the decomposition into irreducible components of the exterior algebra $\bigwedge\left(\mathbb{C}^n \otimes \left(\mathbb{C}^k\right)^*\right)$ regarded as a $GL_n \times GL_k$ module. Irreducible $GL_n \times GL_k$ representations are parameterized by pairs of Young diagrams $(\lambda, \bar{\lambda}')$, where $\bar{\lambda}'$ is the complement conjugate diagram to λ inside the $n \times k$ rectangle. We set the probability of a diagram as a normalized specialization of the character for the corresponding irreducible component. For the principal specialization we get the probability that is equal to the ratio of the q-dimension for the irreducible component over the q-dimension of the exterior algebra. We demonstrate that this probability distribution can be described by the q-Krawtchouk polynomial ensemble. We derive the limit shape and prove the central limit theorem for the fluctuations in the limit when n, k tend to infinity and q tends to one at comparable rates.

Introduction and main results

Various dualities play major role in asymptotic representation theory. In particular, the Schur–Weyl duality between GL_n and $\mathcal{S}_k$ was used by S. Kerov to study the distribution of tensors by symmetry types [16]. If n, k tend to infinity with the same rate the limit shape of Young diagrams in the decomposition of $(\mathbb{C}^n)^{\otimes k}$ into irreducible GL_n-modules coincides with the famous Vershik–Kerov–Logan–Shepp limit shape [28, 20]. This is not the case if $k \sim n^2$, as was demonstrated by P. Biane [3]. The group $\mathcal{S}_k$ is the Weyl group of GL_k therefore the Schur–Weyl duality leads to the (GL_n, GL_k) Howe duality [13, 12], that is the decomposition of the symmetric algebra

$$S\left(\mathbb{C}^n \otimes \mathbb{C}^k\right) \cong \bigoplus_{\ell(\lambda)\leq \min(n,k)} V_{GL_n}(\lambda) \otimes V_{GL_k}(\lambda) \tag{1}$$

into the multiplicity-free and sum of irreducible $GL_n \times GL_k$ modules where the diagrams λ have at most $\min(n,k)$ rows. Restrict the decomposition (1) to the diagrams of at most m columns and consider the probability of a diagram to be proportional to the dimension of an irreducible component. Then this probability measure is the same as the measure on the main diagonal of lozenge tilings of the hexagon with the sides (m,n,k,m,n,k) induced by the uniform measure [7]. The decomposition (1) is also related to celebrated Schur measures [23, 24]. Skew (GL_n, GL_k) Howe duality, that is the multiplicity-free decomposition

$$\bigwedge\left(\mathbb{C}^n \otimes (\mathbb{C}^k)^*\right) \cong \bigoplus_{\lambda} V_{GL_n}(\lambda) \otimes V_{GL_k}(\bar{\lambda}'), \tag{2}$$

is relatively less studied from the probabilistic point of view. The measure on the diagrams λ of size m introduced as the ratio of the dimension of the corresponding irreducible $GL_n \times GL_k$ modules to the dimension of the m-th exterior power was considered in [26]. Nevertheless the relation between the measure that is given by the ratio of the dimension of the irreducible module to the dimension of the whole exterior algebra and the Krawtchouk polynomial ensemble does not appear to be widely known before the paper [22].

The decomposition (2) in terms of the characters is an alternative form of the dual Cauchy identity for Schur polynomials [21]. Therefore we can introduce the probability measure as the ratio of characters

$$\mu_{n,k}(\lambda|\{x_i\}_{i=1}^n, \{y_j\}_{j=1}^k) = \frac{s_\lambda(x_1,\ldots,x_n)s_{\bar{\lambda}'}(y_1,\ldots,y_k)}{\prod_{i=1}^n\prod_{j=1}^k(x_i+y_j)}. \tag{3}$$

This measure (up to a change of $\bar{\lambda}' \to \lambda'$ and $y_j \to 1/y_j$) was considered in [11], but the limit shapes were not explicitly discussed there. We consider the principal specialization of characters of the form $x_i = q^{i-1}, y_j = q^{j-1}$ and the specialization $x_i = q^{i-1}, y_j = q^{1-j}$. As was demonstrated in [22], the measures then take the form

$$\mu_{n,k}(\lambda|q) = \frac{q^{||\lambda||}\dim_q\left(V_{GL_n}(\lambda)\right)\cdot q^{||\bar{\lambda}'||}\dim_q\left(V_{GL_k}(\bar{\lambda}')\right)}{\prod_{i=1}^n\prod_{j=1}^k(q^{i-1}+q^{j-1})}, \tag{4}$$

and

$$\mu_{n,k}(\lambda|q,q^{-1}) = \frac{q^{||\lambda||}\dim_q\left(V_{GL_n}(\lambda)\right)\cdot q^{-||\bar{\lambda}'||}\dim_{1/q}\left(V_{GL_k}(\bar{\lambda}')\right)}{\prod_{i=1}^n\prod_{j=1}^k(q^{i-1}+q^{1-j})}, \tag{5}$$

for each specialization respectively, where $||\lambda|| = \sum_{i=1}^{n}(i-1)\lambda_i$ and q-dimension $\dim_q$ will be defined in Section 1. Our first result is the relation of these measures to q-Krawtchouk ensembles. We recall that normalized q-Krawtchouk orthogonal polynomials $\tilde{K}_j^q(q^{-x}; p, N, q)$ satisfy the following orthogonality relations [18, Section 14.15]

$$\sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_q p^{-i} q^{\binom{i}{2} - iN} \tilde{K}_j^q(q^{-i}; p, N; q) \tilde{K}_l^q(q^{-i}; p, N; q) = \delta_{jl}. \tag{6}$$

Theorem 1 (q-Krawtchouk ensemble)**.** *The probability measure (4) defines a q-Krawtchouk polynomial ensemble,*

$$\mu_{n,k}(\lambda|q) = \det\left(\sqrt{W(a_i)W(a_j)} \sum_{l=0}^{n-1} \tilde{K}_l^q(a_i)\ \tilde{K}_l^q(a_j)\right)_{i,j=1}^{n}, \tag{7}$$

where $a_i = \lambda_i + n - i$ and $\tilde{K}_j^q(x) = \tilde{K}_j^q(q^{-x}; p, N; q)$ are the normalized q-Krawtchouk polynomials with $N = n + k - 1$, $p = q^{1-2n}$, and

$$W(a_i) = q^{\binom{a_i}{2} + a_i(n-k)} \begin{bmatrix} n+k-1 \\ a_i \end{bmatrix}_q.$$

The measure (5) defines a q-Krawtchouk polynomial ensemble for the polynomials $\tilde{K}_j^q(q^{-a_i}; q^{2-2n-k}, n+k-1; q)$.

Next we describe the limit behavior of the correlation kernels of q-Krawtchouk ensembles as $n, k \to \infty$, $q \to 1$ with compatible rates. We prove the convergence of the determinantal point ensembles to the limit determinantal random point process by the method of Borodin and Olshaski [4], which uses the spectral theory of self-adjoint operators on Hilbert space to establish the pointwise convergence of the correlation kernels.

Our derivation of the limit correlation kernel for the q-Krawtchouk ensemble is similar to the proof for the Charlie and Krawtchouk ensembles in [4] and to the proof for the Hahn ensemble in [10], therefore we present only an outline of the proof.

Theorem 2 (Limit correlation kernel)**.** *As $n, k \to \infty$ and $q \to 1$ in such a way that $q = 1 - \frac{\gamma}{n}$ and $\frac{k}{n} \to c$, and the variables a, b are defined as $a = nt + u$, $b = nt + v$, and t, u, v are finite, the correlation kernels*

$$\mathbf{K}_n(a, b) = \sqrt{W(a)W(b)} \sum_{l=0}^{n-1} \tilde{K}_l^q(a)\ \tilde{K}_l^q(b)$$

converge to the discrete sine kernel

$$\lim_{n\to\infty} \mathbf{K}_n(nt+u, nt+v) = \mathbf{K}_\varphi^{sine}(u, v) = \frac{\sin(\varphi(u-v))}{\pi(u-v)}, \tag{8}$$

where for the measure (4) *we have*

$$\varphi = \arccos\left(\operatorname{sgn}(-\gamma)\frac{e^{\gamma-\frac{\gamma t}{2}}}{2}\frac{1-e^{\gamma(c-1)}}{\sqrt{(1-e^{\gamma t})(1-e^{\gamma(c+1-t)})}}\right), \tag{9}$$

and for the measure (5) *we have*

$$\varphi = \arccos\left(\operatorname{sgn}(-\gamma)\frac{e^{\frac{\gamma}{2}(t-c)}}{2}\frac{1-e^{\gamma c}-e^{\gamma(c-t)}+e^{\gamma(c+1-t)}}{\sqrt{(1-e^{\gamma t})(1-e^{\gamma(c+1-t)})}}\right). \tag{10}$$

We also describe the global fluctuations around the average. The change of diagram coordinates from $\{\lambda_i\}_{i=1}^n$ to $\{a_i\}_{i=1}^n$ corresponds to the 45° rotation and then we scale the diagram by the factor $\frac{1}{n}$, switching to coordinates $x_i = \frac{a_i}{n}$. Then the upper boundary of the rotated and scaled diagram defines a continuous piecewise-linear function $f_n \in C([0, c+1])$.

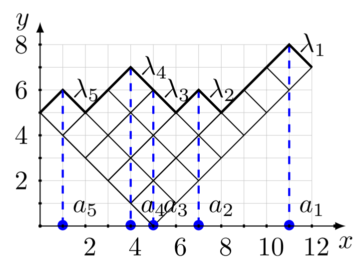

Diagram $\lambda = (7,4,3,3,1)$ is rotated, thick black line is upper boundary f_n, row lengths $\{\lambda_i\}$ correspond to the point positions $\{a_i\}$.

Theorem 3 (Central limit theorem)**.** *Consider random point processes* $\left\{x_i = \frac{a_i}{n}\right\}_{i=1}^n$ *corresponding to the probability distributions (4) or (5). Consider* ρ *given by the formula* (14) *or* (15) *as a function of* $e^{-\gamma t}$ *and denote its support by* $[b-2a, b+2a]$*. Then for a linear statistics* $X_f^{(n)} = \sum_{i=1}^n f(e^{-\gamma x_i})$*, where* $f \in C^1([b-2a, b+2a])$*, we have*

$$X_f^{(n)} - \mathbb{E}X_f^{(n)} \to \mathcal{N}\left(0, \sum_{l\geq 1} l|\widehat{f}_l|^2\right), \tag{11}$$

in distribution, as $n, k \to \infty$ *with* $c = \lim \frac{k}{n}$*, where the Fourier coefficients* $\widehat{f}_l$ *are defined as*

$$\widehat{f}_l = \frac{1}{2\pi}\int_0^{2\pi} f\left(2a\cos\theta + b\right)e^{-il\theta}\mathrm{d}\theta. \tag{12}$$

The values a, b *are given by the formula (56) for the measure (4) and by formula (57) for the measure (5)*

The first correlation function gives us the limit density of points, which is then used to write the explicit expression for the limit shape $f(x)$. We do not present a proof of the uniform convergence of random functions $f_n(x)$ to $f(x)$, since it requires a lot of technical details and therefore will be presented in a separate publication. But one can combine Theorems 2 and 3 to obtain the weak convergence to the limit shape.

Corollary 1. *The limit shape of the upper boundary f_n of a rotated and scaled random Young diagram λ with respect to the probability measure* (4) *is given by the formula*

$$f(x) = 1 + \int_0^x (1 - 2\rho(t)\mathrm{d}t), \tag{13}$$

where the limit density $\rho(t)$ is given by the formula:

$$\begin{aligned}\rho(t) = \lim_{u\to v} \mathbf{K}^{sine}_{\varphi}(u,v) = \frac{\varphi}{\pi} = \\ = \frac{1}{\pi}\arccos\left(\operatorname{sgn}(-\gamma)\frac{e^{\gamma-\frac{\gamma t}{2}}}{2}\frac{1-e^{\gamma(c-1)}}{\sqrt{(1-e^{\gamma t})(1-e^{\gamma(c+1-t)})}}\right)\end{aligned} \tag{14}$$

For the probability measure (5) *the limit density is given by the formula*

$$\rho(t) = \frac{1}{\pi}\arccos\left(\operatorname{sgn}(-\gamma)\frac{e^{\frac{\gamma}{2}(t-c)}}{2}\frac{1-e^{\gamma c}-e^{\gamma(c-t)}+e^{\gamma(c+1-t)}}{\sqrt{(1-e^{\gamma t})(1-e^{\gamma(c+1-t)})}}\right). \tag{15}$$

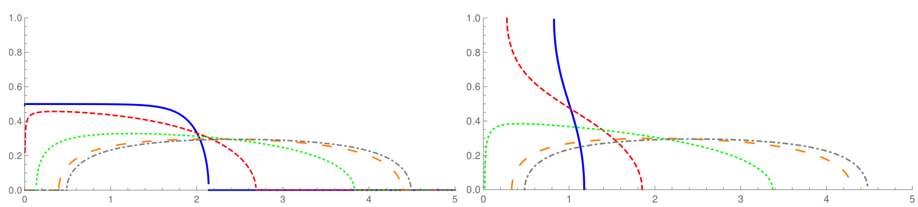

Figure 1: Plots of the limit densities (14) (on the left) and (15) (on the right) for $c = 4$ and the values of γ: -10 (solid blue), -2 (dashed red), -0.5 (dotted green), -0.1 (sparsely dashed orange) -0.01 (dot-dashed gray).

Plots of the densities (14) and (15) for $c = 4$ are presented for various values of γ in Fig. 1, and the corresponding limit shapes are presented in Fig. 2.

The paper is organized as follows. In Section 1 we use the explicit formulas for the q-dimensions to prove Theorem 1. In Section 2 we derive the limit shapes and outline the proof of Theorem 2. Then in Section 3 we discuss the fluctuations and prove the central limit theorem. We discuss some open questions in the conclusion.

1 q-Krawtchouk polynomial ensemble

In this section we prove Theorem 1. We first recall the derivation of the explicit formula for the measures (4), (5) from [22, Theorem 4.6] and then use it to demonstrate (7).

It is well known that the GL_n character, which is the Schur polynomial $s_\lambda(x_1,\ldots,x_n)$, is given by the following sum over the semi-standard Young

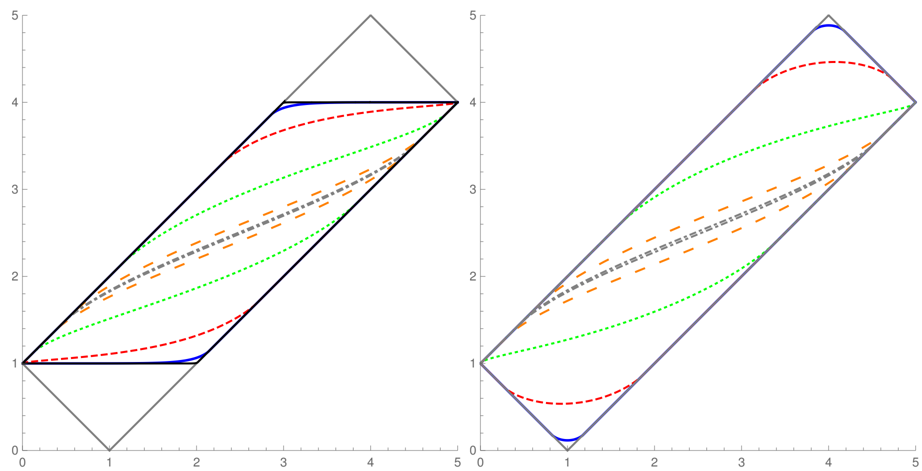

Figure 2: Plots of the limit shapes for Young diagrams corresponding to the densities (14) (on the left) and (15) (on the right) for $c = 4$ and the values of γ (bottom to top): -10 (solid blue), -2 (dashed red), -0.5 (dotted green), -0.1 (sparsely dashed orange), -0.01 (dot-dashed gray), 0.01 (dot-dashed gray), 0.1 (sparsely dashed orange), 0.5 (dotted green), 2 (dashed red), 10 (solid blue). Solid black lines on the left panel correspond to $\gamma = \pm\infty$ ($q = \text{const}$).

tableaux of the shape λ:

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda,n)} \prod_{i=1}^{n} x_i^{\#i's \text{ in } T}. \tag{16}$$

Define q-dimension of the irreducible GL_n representation as the principal gradation (see [15, §10.10]) that is the weighted sum of the dimensions of weight subspaces:

$$\dim_q \left(V_{GL_n}(\lambda)\right) = \sum_{(u_1,\dots,u_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}} q^{\sum_{i=1}^{n-1} u_i} \dim V(\lambda)_{\lambda - \sum_{i=1}^{n-1} u_i \alpha_i}, \tag{17}$$

where $\alpha_1, \dots, \alpha_{n-1}$ are the simple roots of GL_n and we identify the diagram λ with the GL_n weight λ. We use the notation

$$[m]_q = \frac{1 - q^m}{1 - q} \tag{18}$$

for the q-numbers, define q-factorials as products of q-numbers and q-binomial coefficients as the ratio of q-factorials. The formulas for q-Krawtchouk polynomials in [18, Section 14.15] use q-Pochhammer symbols, defined as

$$(a;q)_0 = 1, \quad (a;q)_m = \prod_{i=1}^{m} (1 - aq^{i-1}), \; m \in \mathbb{Z}_+. \tag{19}$$

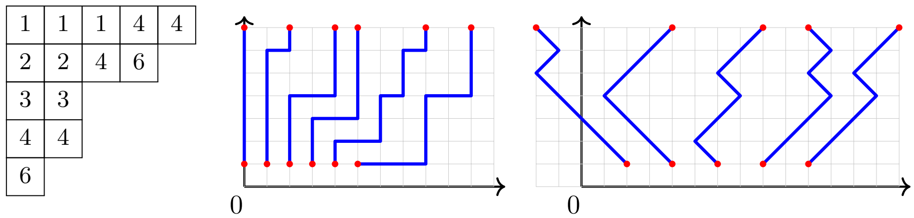

Figure 3: A GL_6-Young tableau, its row reading (*left*) and its column readings (*right*).

By the standard row-reading rule where the number of i-boxes in the row j corresponds to the number of horizontal steps on the level i in the path number j (see Fig. 3 (*left*)), semistandard Young tableaux $SSYT(\lambda, n)$ are in one-to-one correspondence with the configurations of n non-intersecting paths that start at points $(0,1),(1,1),\ldots,(n-1,1)$ and end at points $(a_n,n),\ldots,(a_1,n)$. Horizontal steps on the level i are weighted by $x_i = q^{i-1}$ and vertical steps have weight 1. Using the Lindström–Gessel–Viennot lemma [8, 19] and the recursion on the determinants it is easy to derive a well-known q-analog of the Weyl dimension formula [17, 5]:

$$s_\lambda(1,q,q^2,\ldots,q^{n-1}) = q^{||\lambda||}\dim_q\left(V_{GL_n}(\lambda)\right) = q^{||\lambda||}\prod_{i=1}^{n}\prod_{j=i+1}^{n}\frac{[a_i-a_j]_q}{[j-i]_q}. \quad (20)$$

Similarly we get the formula for the q-dimension of the GL_k-representation $V_{GL_k}(\bar{\lambda}')$, but we would like to have it in terms of row lengths of λ, not $\bar{\lambda}'$. Therefore we use the column reading for the bijection between non-intersecting lattice paths and semistandard Young tableaux. For the semistandard Young tableau $T \in SSYT(\mu, k)$ of at most n columns of lengths $\mu'_1,\ldots,\mu'_n$ the paths start at points $(2,1),(4,1),\ldots(2n,1)$ and go to the points $(2i-2\mu'_i+k, k+1)$ for $i=1,\ldots,n$. The paths consists of k steps $(-1,1)$ or $(1,1)$ and the step number j is $(-1,1)$ if j is present in the column (see Fig. 3 (*right*)). We weight the $(-1,1)$ steps by $q^{m/2}$ where m is the number of the left-leaning diagonal starting from 0. We again can use Lindström–Gessel–Viennot lemma and Dodgson condensation (see, e.g. [5]) for the determinants to obtain the formula

$$s_{\bar{\lambda}'}(1,q,q^2,\ldots,q^{k-1}) = q^{||\bar{\lambda}'||}\dim_q\left(V_{GL_k}(\bar{\lambda}')\right) = \\ = q^{||\bar{\lambda}'||}\prod_{1\le i<j\le n}^{n}[a_i-a_j]_q\cdot\prod_{l=1}^{n}\frac{[n+k-l]_q!}{[a_l]_q![n+k-1-a_l]_q!}. \quad (21)$$

Now we can rewrite the measure (4) as

$$\mu_{n,k}(\lambda|q) = \frac{q^{\|\lambda\|+\|\bar{\lambda}'\|}}{\prod_{i=1}^{n}\prod_{j=1}^{k}(q^{i-1}+q^{j-1})} \times \frac{\prod_{m=0}^{n-1}[k+m]_q!\prod_{1\le i<j\le n}[a_i-a_j]_q^2}{\prod_{i<j}[j-i]_q\prod_{i=1}^{n}[a_i]_q![n+k-1-a_i]_q!}, \tag{22}$$

where

$$\|\bar{\lambda}'\| = \sum_{i=1}^{n}\frac{(k-\lambda_i)(k-\lambda_i-1)}{2} = \sum_{i=1}^{n}\binom{n+k-a_i-i}{2}. \tag{23}$$

We rewrite the power of q as

$$q^{\|\lambda\|+\|\bar{\lambda}'\|} = q^{\sum_{i=1}^{n}(i-1)(a_i-n+i)+\binom{n+k-a_i-i}{2}} \propto q^{\sum_{i=1}^{n}\binom{a_i}{2}+2a_i(i-n)+a_i(n-k)}, \tag{24}$$

and q-analog of the Vandermonde determinant as

$$\begin{aligned}\prod_{1\le i<j\le n}[a_i-a_j]_q^2 &\propto \prod_{1\le i<j\le n}(1-q^{a_i-a_j})^2 = \\ &= \prod_{1\le i<j\le n}q^{2a_i}(q^{-a_i}-q^{-a_j})^2 = q^{\sum_{i=1}^{n}2a_i(n-i)}\prod_{i<j}(q^{-a_i}-q^{-a_j})^2,\end{aligned} \tag{25}$$

to write the measure in the form of a determinantal ensemble

$$\mu_{n,k}(\{a_i\},q) = C_{n,k,q}\prod_{i<j}(q^{-a_i}-q^{-a_j})^2\prod_{i=1}^{n}W(a_i), \tag{26}$$

where

$$W(a_i) = q^{\binom{a_i}{2}+a_i(n-k)}\begin{bmatrix}n+k-1\\a_i\end{bmatrix}_q \tag{27}$$

and

$$C_{n,k,q} = \frac{q^{\frac{kn}{2}(n+k-2)}}{\prod_{i=1}^{n}\prod_{j=1}^{k}(q^{i-1}+q^{j-1})}\prod_{i=1}^{n}\frac{[k+i-1]_q!}{[i-1]_q!\,[n+k-1]_q!}\frac{1}{(1-q)^{\frac{n(n-1)}{2}}}. \tag{28}$$

The weight $W(a_i)$ coincides [(14.15.2) 18] with the weight for q-Krawtchouk polynomials $K_m^q(q^{-x};p,N;q)$ with the parameters $p=q^{1-2n}$ and $N=n+k-1$. Therefore the equality (7) is proven.

Similarly, the measure $\mu(\lambda|,q,q^{-1})$ is written explicitly as

$$\mu_{n,k}(\lambda|q,q^{-1}) = \hat{C}_{n,k,q}\prod_{i<j}(q^{-a_i}-q^{-a_j})^2\prod_{i=1}^{n}\hat{W}(a_i), \tag{29}$$

where

$$\hat{W}(a_i) = q^{\binom{a_i}{2}+a_i(n-1)}\begin{bmatrix}n+k-1\\a_i\end{bmatrix}_q \tag{30}$$

and

$$\hat{C}_{n,k,q} = \frac{q^{\frac{n}{2}(n-1)(n+2k-2)}}{\prod_{i=1}^{n}\prod_{j=1}^{k}(q^{i-1}+q^{1-j})}\prod_{i=1}^{n}\frac{[k+i-1]_q!}{[i-1]_q!\,[n+k-1]_q!}\frac{1}{(1-q)^{\frac{n(n-1)}{2}}}. \tag{31}$$

Taking $p=q^{2-2n-k}$ and $N=n+k-1$ we conclude the proof of Theorem 1.

2 Correlation kernels and limit density

In this section we outline the proof of Theorem 2. As was shown in the previous section, the upper boundary of a rotated random Young diagram corresponds to a point configuration. Therefore to derive the limit shape it is sufficient to find the limit density of the points. We use the q-difference equation for q-Krawtchouk polynomials to derive the limit density by the method of Borodin and Olshanski [4]. The limit density is given by the discrete sine-kernel as one expects from its universal properties [2].

For any n-point discrete determinantal polynomial ensemble $\mathcal{P}^{(n)}$ with the weight function $W^{(n)}(x)$ and normalized orthogonal polynomials $p_i^{(n)}(x)$ defined on a finite lattice $\{x_0^{(n)}, \dots, x_L^{(n)}\}$, m-point correlation function can be written as a determinant

$$\rho_m^{(n)}(x_1, \dots, x_m) = \det[\mathbf{K}_n(x_i, x_j)]_{1\le i,j\le m}, \tag{32}$$

where $\mathbf{K}_n(x_i, x_j)$ is the correlation kernel defined by the formula

$$\mathbf{K}_n(x, y) = \sum_{i=0}^{n-1} \sqrt{W^{(n)}(x)W^{(n)}(y)}\; p_i^{(n)}(x)p_i^{(n)}(y). \tag{33}$$

The variables x, y take values on the lattice $\{x_0^{(n)}, \dots, x_L^{(n)}\}$, therefore we can consider the polynomials and the weights $W^{(n)}$ to be the functions of the corresponding integer index $p_i^{(n)}(x_l) = p_i^{(n)}(l)$. Then the functions $\{\sqrt{W^{(n)}(x)}\, p_i^{(n)}(x)\}_{i=0}^{L}$ form an orthonormal basis in the space ℓ^2 on the finite set $\{0, \dots, L\}$. The correlation kernel acts on this space by projecting to the subspace spanned by the first n states $\{\sqrt{W^{(n)}(x)}\; p_i^{(n)}(x)\}_{i=0}^{n-1}$. To prove the convergence of the determinantal ensembles $\{\mathcal{P}^{(n)}\}$ as $n\to\infty$ to a determinantal point process, it is sufficient to demonstrate the pointwise convergence of the correlation kernels with an appropriate scaling of the arguments $\mathbf{K}_n(nt + x, nt + y) \xrightarrow[n\to\infty]{} \mathbf{K}_t(x, y)$ [4, 1]. Assume that as $n \to \infty$, the lattice also grows, so $L \to \infty$ as well and its points fill some interval. Then the limit density of points in the point ensemble can be recovered from the 1-point function $\rho(t|y) = \lim_{x\to y} \mathbf{K}_t(x, y)$.

Consider the continuation of the functions $W^{(n)}, p_i^{(n)}$ to $\mathbb{Z}_+$ by assuming zero values at x_i for $i > L$. Assume that for any n there exists a bounded self-adjoint operator $D^{(n)}$ in $\ell^2(\mathbb{Z}_+)$ such that $\{\sqrt{W^{(n)}(x)}\; p_i^{(n)}(x)\}_{i=0}^{L}$ are its eigenfunctions and assume that there exits a limit $D^{(n)} \xrightarrow[n\to\infty]{} D$ in strong resolvent sense, where D is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}_+)$ with a simple continuous spectrum $[\alpha, \beta]$. Then the theorem VIII.24 in [27] implies the convergence of the corresponding spectral projections.

The limit correlation kernel is given by a spectral projection to a subinterval of $[\alpha, \beta]$. Moreover, Hilbert space $\ell^2(\mathbb{Z}_+)$ is isomorphic to $L^2([\alpha, \beta], d\nu)$, where $d\nu$ is the spectral measure on $[\alpha, \beta]$. The operator D on $L^2([\alpha, \beta], d\nu)$ becomes a multiplication operator and spectral projection is given by the characteristic function. Taking the Fourier transform from L^2 to ℓ^2, we can recover the limit correlation kernel $\mathbf{K}(x,y)$.

In our case we consider the ensemble of q-Krawtchouk polynomials, defined on the lattice $q^{-a} = e^{\gamma\frac{a}{n}}$, $a = 0,\dots,n+k-1$ with the weight (27) or (30). Since number of lattice points grows with n, we need to rescale our problem and consider random variables $x = \frac{a}{n}$ that take values on the interval $[0, c+1)$. Recall that q-difference equation for the q-Krawtchouk polynomials $K_m^q(q^{-a}; p, N; q) = K_m^q(a)$ is given by [(14.15.5) 18]

$$A(m)K_m^q(a) = B(a)K_m^q(a+1) - (B(a)+C(a))K_m^q(a) + C(a)K_m^q(a-1), \quad (34)$$

where we have omitted some arguments of the coefficients A, B, C for brevity:

$$A(m) = A(m,q) = q^{-m}(1-q^m)(1+pq^m), \quad (35)$$

$$B(a) = B(q,a,N) = 1 - q^{a-N}, \quad (36)$$

$$C(a) = C(q,a) = -p(1-q^a). \quad (37)$$

Rewriting for the functions $\kappa_m(a) = \sqrt{W(a)}\tilde{K}_m^q(a)$ and canceling the normalization constant, we obtain

$$\frac{A(m)}{\sqrt{W(a)}}\kappa_m(a) = \frac{B(a)}{\sqrt{W(a+1)}}\kappa_m(a+1) - \frac{(B(a)+C(a))}{\sqrt{W(a)}}\kappa_m(a) + \frac{C(a)}{\sqrt{W(a-1)}}\kappa_m(a-1). \quad (38)$$

Then we move some terms to the other side and multiply both sides by $\sqrt{W(a)}$ to get

$$B(a)\sqrt{\frac{W(a)}{W(a+1)}}\kappa_m(a+1) + C(a)\sqrt{\frac{W(a)}{W(a-1)}}\kappa_m(a-1) = \\ = (A(m)+B(a)+C(a))\kappa_m(a). \quad (39)$$

If we express $W(a+1)$ as a product of $W(a)$ and the remaining term,

$$W(a+1) = W(a)q^{a+1-N}p^{-1}\frac{[N-a]_q}{[a+1]_q} = W(a)q^{a+1-N}p^{-1}\frac{1-q^{N-a}}{1-q^{a+1}}, \quad (40)$$

$$W(a-1) = W(a)q^{N-a}p\frac{1-q^a}{1-q^{N+1-a}}, \quad (41)$$

it is easy to check that

$$B(a)\frac{\sqrt{W(a)}}{\sqrt{W(a+1)}} = C(a+1)\frac{\sqrt{W(a+1)}}{\sqrt{W(a)}} = -\sqrt{pq^{a-N+1}(1-q^{a+1})(1-q^{N-a})}. \quad (42)$$

Now the left hand side of (39) can be seen as an action of an operator $D^{(n)}$ in $\ell^2(\mathbb{Z}_+)$ on its eigenfunction $\kappa_m(a)$. This action can be seen as a convolution with the matrix $D^{(n)}(a,b)$: $(D^{(n)}f)(a) = \sum_{b=0}^{\infty} D^{(n)}(a,b)f(b)$. In the natural $\ell^2(\mathbb{Z}_+)$ basis $\{\delta_i\}_{i=0}^{\infty}$ the matrix elements are

$$D^{(n)}(i,j) = \begin{cases} B(i)\sqrt{\frac{W(i)}{W(i+1)}}, & j=i+1, \quad i,j \le L, \\ C(i)\sqrt{\frac{W(i)}{W(i-1)}}, & j=i-1, \quad i,j \le L, \\ 1, & i=j, \quad i,j > L \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Clearly, the operator $D^{(n)}$ is self-adjoint. Therefore by using Theorem VIII.25 in [27], similarly to Hahn ensemble in [10], we get the convergence to the limit operator D, defined by the corresponding three-diagonal Jacobi matrix, in the strong resolvent sense.

It is easy to check that in the limit $N, a \to \infty, a \sim N$ the ratio of the coefficients on the left hand side of equation (39) converges to 1

$$\frac{B(a)}{C(a)}\sqrt{\frac{W(a-1)}{W(a+1)}} = \\ = -\frac{1-q^{a-N}}{1-q^a}p^{-1}\sqrt{\frac{(1-q^a)(1-q^{a+1})}{(1-q^{N-a})(1-q^{N+1-a})}p^2q^{2(N-a)-1}} \xrightarrow[N,a\to\infty]{} 1. \tag{44}$$

Therefore for the ease of computation we can rewrite the difference equation in the form

$$\frac{B(a)}{C(a)}\sqrt{\frac{W(a-1)}{W(a+1)}}\kappa_m(a+1)+\kappa_m(a-1) = \sqrt{\frac{W(a+1)}{W(a)}}\left(\frac{A(m)}{B(a)}+1+\frac{C(a)}{B(a)}\right)\kappa_m(a). \tag{45}$$

Then eigenvalues on the right hand side are:

$$q^{\frac{1}{2}(a-N)}p^{-\frac{1}{2}}\sqrt{\frac{[N-a]_q}{[a+1]_q}}\left(\frac{q^{-m}(1-q^m)(1+pq^m)}{1-q^{a-N}}+1-\frac{p(1-q^a)}{(1-q^{a-N})}\right), \tag{46}$$

where $m = 0,\ldots,N$. Substituting $p = q^{1-2n}$, $N = n+k-1$, $q = e^{-\gamma\frac{1}{n}}$, $a = nx$ and taking the limit $n, k \to \infty$, we see that eigenvalues fill the interval

$$\Bigg[e^{\frac{\gamma}{2}(c+1-x)-\gamma}\sqrt{\frac{1-e^{-\gamma(c+1-x)}}{1-e^{-\gamma x}}}\left(\frac{(e^{(c+1)\gamma}+1)(e^{(c+1)\gamma}-1)}{1-e^{\gamma(c+1-x)}}+1-\frac{e^{2\gamma}(1-e^{-\gamma x})}{1-e^{\gamma(c+1-x)}}\right), \\ e^{\frac{\gamma}{2}(c+1-x)-\gamma}\sqrt{\frac{1-e^{-\gamma(c+1-x)}}{1-e^{-\gamma x}}}\left(1-\frac{e^{2\gamma}(1-e^{-\gamma x})}{1-e^{\gamma(c+1-x)}}\right)\Bigg]. \tag{47}$$

The corresponding limit operator $\widetilde{D}$ then acts as a difference operator

$$\widetilde{D}f(x) = f(x+1)+f(x-1). \tag{48}$$

The operator $\widetilde{D}$ is self-adjoint and has simple purely continuous Lebesgues spectrum. The correlation kernel $\mathbf{K}_n(a,b)$ is the projection to the part of the spectrum that in the limit becomes

$$\left[e^{\frac{\gamma}{2}(c+1-x)-\gamma}\sqrt{\frac{1-e^{-\gamma(c+1-x)}}{1-e^{-\gamma x}}}\left(\frac{(e^{\gamma}+1)(e^{\gamma}-1)}{1-e^{\gamma(c+1-x)}}+1-\frac{e^{2\gamma}(1-e^{-\gamma x})}{1-e^{\gamma(c+1-x)}}\right),\right.$$
$$\left. e^{\frac{\gamma}{2}(c+1-x)-\gamma}\sqrt{\frac{1-e^{-\gamma(c+1-x)}}{1-e^{-\gamma x}}}\left(1-\frac{e^{2\gamma}(1-e^{-\gamma x})}{1-e^{\gamma(c+1-x)}}\right)\right]. \quad (49)$$

This spectral projection is given by the discrete sine kernel

$$\mathbf{K}^{sine}_{\varphi}(u,v)=\frac{\sin(\varphi(u-v))}{\pi(u-v)}, \quad (50)$$

since the Fourier transform of the difference operator $\widetilde{D}$ from $\ell^2(\mathbb{Z})$ to L^2 on unit circle $|z|=1$ is the multiplication by the function $z+\bar{z}=2\Re z$ and has purely continuous (double) spectrum $[-2,2]$. The maximum of the interval (49) is always ≥ 2 while the minimum is inside of $[-2,2]$. So in the L^2 space the spectral projection is the multiplication by the characteristic function of the arc from $e^{-i\varphi}$ to $e^{i\varphi}$. In the ℓ^2 realization, this is the integral operator with the discrete sine kernel with φ given by the formula

$$\varphi=\arccos\left(\frac{1}{2}e^{\frac{\gamma}{2}(c+1-x)-\gamma}\sqrt{\frac{1-e^{-\gamma(c+1-x)}}{1-e^{-\gamma x}}}\left(\frac{(e^{\gamma}+1)(e^{\gamma}-1)}{1-e^{\gamma(c+1-x)}}+1-\frac{e^{2\gamma}(1-e^{-\gamma x})}{1-e^{\gamma(c+1-x)}}\right)\right). \quad (51)$$

Now as we are interested in the one-point correlation function, we take limit $u\to v$ in the correlation kernel and get $\rho(x)=\frac{\varphi}{\pi}$, which after a simplification becomes (14). Similarly, taking $N=n+k-1$ and $p=q^{2-2n-k}$ and weight (30), we recover the spectral interval and the limit density (15) for the measure (5).

Since the points a_i of the discrete q-Krawtchouk ensemble correspond to the intervals where the upper boundary f_n decays, the density of these points is connected to the derivative of f_n by the formula $f_n'(x)=1-2\rho_1^{(n)}(x)$. Thus from the limit density we recover the limit shape by the formula (13).

3 Fluctuations

In this section we prove Theorem 3 applying the general approach of Breuer and Duits [6]. Due to their result, it is sufficient to establish the convergence of the coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials. Then we can apply the following theorem.

Theorem 4 (Theorem 2.5 in [6]). *Let $\{p_m^{(n)}(x)\}_{m=0}^{n-1}$ be normalized orthogonal polynomials of the polynomial ensemble $\mathcal{P}_n$ satisfying the three-term recurrence* *relation*

$$xp_m^{(n)}(x) = a_{m+1}^{(n)} p_{m+1}^{(n)}(x) + b_{m+1}^{(n)} p_m^{(n)}(x) + a_m^{(n)} p_{m-1}^{(n)}(x),$$

and assume that there exists a subsequence $\{n_j\}_j$ *and* $a > 0, b \in \mathbb{R}$ *such that for any* $k \in \mathbb{Z}$ *we have*

$$a_{n_j+k}^{(n_j)} \to a, \quad b_{n_j+k}^{(n_j)} \to b,$$

as $j \to \infty$*. Then for any real-valued* $f \in C^1(\mathbb{R})$ *we have*

$$X_f^{\mathcal{P}_{n_j}} - \mathbb{E} X_f^{\mathcal{P}_{n_j}} \to \mathcal{N}\left(0, \sum_{l \geq 1} l |\widehat{f}_l|^2\right), \quad \text{as } j \to \infty,$$

in distribution, where the coefficients $\widehat{f}_l$ *are defined as*

$$\widehat{f}_l = \frac{1}{2\pi} \int_0^{2\pi} f(2a\cos\theta + b) e^{-il\theta} d\theta,$$

for $l \geq 1$*. When* $n_j = j$*, that is the subsequence is the whole sequence,* (4) *is equivalent to*

$$a_n^{(n)} \to a, \quad b_n^{(n)} \to b.$$

Similarly to the study of the lozenge tilings and Hahn polynomial ensemble in [6, Section 6.2], we establish the required convergence for the normalized q-Krawtchouk polynomials. We start with the recurrence relation for the monic q-Krawtchouk polynomials $P_n(q^{-x})$ [18, formula 14.15.4]:

$$q^{-x} P_n(q^{-x}) = P_{n+1}(q^{-x}) + [1 - (A_n + C_n)] P_n(q^{-x}) + A_{n-1} C_n P_{n-1}(q^{-x}),$$

where $P_n(q^{-x}) = \frac{(q^{-N};q)_n}{(-pq^n;q)_n} K_n(q^{-x}; p, N; q)$, q-Pochhammer symbols are defined by the formula (19) and

$$A_n = \frac{(1 - q^{n-N})(1 + pq^n)}{(1 + pq^{2n})(1 + pq^{2n+1})}, \tag{52}$$

$$C_n = -pq^{2n-N-1} \frac{(1 + pq^{n+N})(1 - q^n)}{(1 + pq^{2n-1})(1 + pq^{2n})}. \tag{53}$$

If the monic polynomials $P_n(x)$ satisfy the recursion relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1},$$

then the normalized polynomials $p_n(x)$ satisfy the recursion relation (4) with $a_n = \sqrt{\beta_n}$, $b_{n+1} = \alpha_n$ [14]. Therefore for the normalized q-Krawtchouk polynomials we have

$$a_n = \sqrt{A_{n-1} C_n}, \tag{54}$$

$$b_{n+1} = 1 - A_n - C_n. \tag{55}$$

Substituting $p = q^{1-2n}, N = n + k - 1, q = e^{-\gamma\frac{1}{n}}$ and taking the limit $n \to \infty$, we get

$$\begin{aligned} a = \lim_{n\to\infty} a_n &= \tfrac{1}{4}\sqrt{(e^{\gamma}-1)(e^{\gamma}+1)(1-e^{c\gamma})(1+e^{c\gamma})}, \\ b = \lim_{n\to\infty} b_n &= \tfrac{1+e^{\gamma(c+1)}}{2}. \end{aligned} \tag{56}$$

The interval $[b-2a, b+2a]$ is exactly the support of the limit density (14), written in terms of the variable e^t. This can be easily verified by solving the equation

$$\mathrm{sgn}(-\gamma)\frac{e^{\gamma-\frac{\gamma t}{2}}}{2}\frac{1-e^{\gamma(c-1)}}{\sqrt{(1-e^{\gamma t})(1-e^{\gamma(c+1-t)})}} = \pm 1.$$

Now we apply Theorem 4 to get the desired result for the probability (4).

Similarly, substituting $N = n + k - 1$, $q = e^{-\gamma\frac{1}{n}}$ and $p = q^{2-2n-k}$ into (52)–(55), we obtain

$$\begin{aligned} a = \lim_{n\to\infty} a_n &= \tfrac{e^{c\gamma}}{(1+e^{c\gamma})^2}\sqrt{2(e^{\gamma}-1)(e^{c\gamma}-1)(1-e^{c\gamma})(1+e^{\gamma(c+1)})}, \\ b = \lim_{n\to\infty} b_n &= \tfrac{3e^{\gamma(c+2)}-e^{\gamma(c+1)}+3e^{c\gamma}-e^{2c\gamma}}{(1+e^{c\gamma})^2}. \end{aligned} \tag{57}$$

The interval $[b-2a, b+2a]$ is exactly the support of the limit density (15), written in terms of the variable e^t. This can be easily verified by solving the equation

$$\mathrm{sign}(-\gamma)\frac{e^{\frac{\gamma}{2}(t-c)}}{2}\frac{1-e^{\gamma c}-e^{\gamma(c-t)}+e^{\gamma(c+1-t)}}{\sqrt{(1-e^{\gamma t})(1-e^{\gamma(c+1-t)})}} = \pm 1.$$

Again, we apply Theorem 4 to finish the proof of Theorem 3.

Conclusion and outlook

In the present paper we have discussed the q-Krawtchouk polynomial ensemble from the point of view of the skew Howe duality. We have connected the limit densities of the point processes to the limit shapes of random Young diagrams that parameterize the decomposition of the exterior algebra $\bigwedge\left(\mathbb{C}^n \otimes \left(\mathbb{C}^k\right)^*\right)$ into the irreducible $GL_n \times GL_k$-modules with respect to the principal-principal and principal-inverse principal specializations of the character measure (3). As was demonstrated in the paper [22], such measures can be thought of as the measures induced on the main diagonal of the lozenge tiling for the skew-glued hexagon from the tilings weighted by $q^{\text{Volume of boxes}}$. Volume-weighted tilings of the hexagon can be related to the symmetric (GL_n, GL_k) Howe duality (See [7, 23, 25, 9]). It remains an interesting question to describe the limit surface of such volume-weighted skew-glued tilings that represent a skew Howe duality counterpart to the ordinary volume-weighted tilings of the hexagon.

In Fig. 4 we present a random lozenge tiling of a skew glued hexagon that contributes to a random Young diagram λ chosen from $\mu_{n,k}(\lambda|q)$, with $n = 20$, $k = 80$, $\gamma = -0.5$, that corresponds to the positions of the right triangles on the main diagonal. The complement conjugate diagram $\bar{\lambda}'$ corresponds to the positions of the left triangles on the main diagonal.

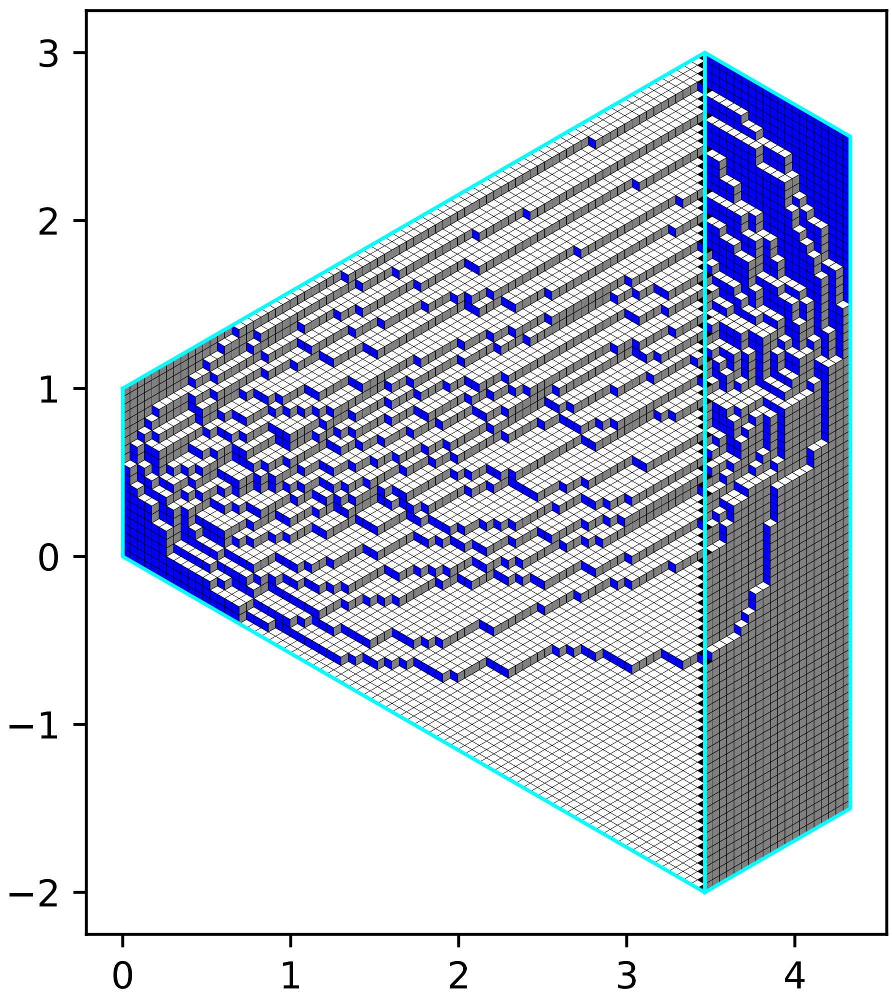

Figure 4: Lozenge tiling of skew glued hexagon, Young diagrams $\lambda, \bar{\lambda}'$ correspond to the positions of right (left) triangles on the main diagonal.

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