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Existence of an Invariant Foliation Near a Locally Integral Surface of Neutral Type

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Abstract—We consider a system of essentially nonlinear differential equations that does not have linear terms on the right-hand side in a neighbourhood of the rest point. Earlier, for this system, the author proved the existence of two locally integral surfaces of so-called "stable" and "neutral" types. In this article, we prove the existence of a foliation into surfaces of stable type in some neighborhood of a neutral surface under the additional assumption that the zero solution on this surface is Lyapunov uniformly stable. This result generalizes the well-known one for quasilinear systems of ODEs. Instead of assumptions on the eigenvalues of the linear approximation, we use conditions on the logarithmic norms of the Jacobi matrices of the right-hand sides. The result obtained is important for describing the behavior of integral curves of complicated systems in a neighborhood of a stationary point, for the theory of stability of solutions, for local equivalence of ODEs.

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1. INTRODUCTION

Consider a system of essentially nonlinear differential equations

$$\dot{x} = X(t,z), \quad \dot{y} = G(y) + F(x) + Y(t,z), \tag{1}$$

where $z = (x,y) \in \mathbb{R}^p \times \mathbb{R}^q$. We assume that this system satisfies the following assumptions:

1) The vector functions X, G, F, and Y are continuous in their arguments and continuously differentiable with respect to x and y for $||z|| \leqslant a, t \in \mathbb{R}$. For all $t \in \mathbb{R}$

$$X(t,0) = 0, \quad G(0) = F(0) = Y(t,0) = 0$$

(i.e. $x = 0, y = 0$ is a solution to the system (1)).

2) $G(y)$ is a homogeneous function of degree $k > 1$, where $(-1)^k = -1$ (wherein k may be rational), and there is a $b > 0$ such that

$$\gamma^*(G'(y)) \leqslant -b||y||^{k-1}, \tag{2}$$

where γ^* denotes the so-called *upper logarithmic norm* generated by a given vector norm $||y||$ in $\mathbb{R}^q$ (see below for logarithmic norms and their properties).

3) $F(x)$ is also a homogeneous function of degree $m > k$ and there exist constants $r_1 > 0$, $r_2 > 0$, such that

$$||F(x)|| \geqslant r_1||x||^m, \quad ||F'(x)|| \leqslant r_2||x||^{m-1}, \tag{3}$$

where $||x||$ is some norm in $\mathbb{R}^p$.

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4) Finally, for $||z|| \leqslant a, t \in \mathbb{R}$, the functions X and Y satisfy the following inequalities:

$$||X_z'(t,z)||, ||Y_z'(t,z)|| \leqslant C_1||z||^{m-1}, \tag{4}$$

$$||Y(t,x,0)|| \leqslant C_2||x||^{m+1}, \tag{5}$$

where $C_1, C_2 \geqslant 0$.

In this paper we choose norms $||x||$ and $||y||$ in such a way that conditions (2)–(5) are satisfied. As for $||z||$, by definition we put

$$||z|| \stackrel{\text{def}}{=} \max(||x||, ||y||). \tag{6}$$

The cases where system (1) is periodic with respect to t or autonomous are also treated in this article.

2. LOGARITHMIC NORMS

Definition 1. The number

$$\gamma^*(A) \stackrel{\text{def}}{=} \lim_{h\to+0} \frac{||E+hA||-1}{h},$$

where E denotes the identity matrix $(q \times q)$, is called the *upper logarithmic norm* of a $(q \times q)$ matrix A.

The logarithmic norm is generated by a given vector norm $||y||$ in $\mathbb{R}^q$. In the right-hand side of the definition of γ^* we use the operator matrix norm generated by the norm $||y||$.

Logarithmic norms were introduced by Lozinsky [1] and by Dahlquist [2] independently in order to estimate the errors of numerical integration of differential equations. Lozinsky also proved some estimates for the characteristic exponents of linear systems in terms of logarithmic norms. This led to the widespread use of logarithmic norms in the theory of linear systems. However, logarithmic norms can also be effectively applied to essentially nonlinear systems [3]. This application was developed and demonstrated in papers [4–8]. We shall only note that logarithmic norms play the same role for essentially nonlinear systems as spectral estimates for quasilinear systems. In this article, we need some properties of logarithmic norms. For convenience, these properties are listed below (see proofs in [1, 3]):

1) $\gamma^*(A+B) \leqslant \gamma^*(A) + \gamma^*(B)$;

2) $\gamma^*(\alpha A) = \alpha\gamma^*(A), \alpha \geqslant 0$;

3) $|\gamma^*(A) - \gamma^*(B)| \leqslant \|A-B\|$;

4) Let a continuous matrix function $A(\theta)$ be given on a finite interval $\langle a, b\rangle$ and suppose that $\gamma^*(A(\theta)) \leqslant 0$. Then $\gamma^*\left(\int\limits_a^b A(\theta)d\theta\right) \leq \int_a^b \gamma^*(A(\theta))d\theta$.

Lower logarithmic norm is introduced alongside the *upper* one by the formula $\gamma_*(A) \stackrel{\text{def}}{=} -\gamma^*(-A)$. The corresponding properties of γ_* can be easily obtained from this definition. The key role of γ_* and γ^* is revealed in the following theorem [3]:

Theorem 1. *Let $y(t)$ be an arbitrary solution of the system $\dot{y} = F(t,y)y + Y(t,y)$, where F is a continuous matrix function and Y is a continuous vector function. Then*

$$\gamma_*\left(F(t,y(t))\right)||y(t)|| - ||Y(t,y(t))|| \leqslant \frac{d_+||y(t)||}{dt} \leqslant \gamma^*\left(F(t,y(t))\right)||y(t)|| + ||Y(t,y(t))||.$$

In this theorem, $||y||$ is an arbitrary vector norm, γ_* and γ^* are the lower and upper logarithmic norms generated by this vector norm, and $d_+||y(t)||/dt$ is the right-hand derivative of the norm of the solution (we shall note that this derivative always exists since the norm is a convex function). For information about one-sided derivatives, differential inequalities, and other general questions of the theory of differential equations, we refer to [9].

3. FORMULATION OF THE MAIN THEOREM

In the paper [4] it is proven that system (1) has a locally integral surface of "stable" type. This surface is defined by the equation $x = h(t, y)$, where h has the following properties:

1) $h \in C(\mathbb{R} \times \{||y|| \leqslant a\} \mapsto \mathbb{R}^p)$;

2) $h(t, 0) = 0\, \forall t \in \mathbb{R}$;

3) $\forall L \in (0, 1]\, \exists a(L) > 0 : \forall ||y_1||, ||y_2|| \leqslant a(L)$ and $\forall t \in \mathbb{R}$ the inequality
$$||h(t, y_1) - h(t, y_2)|| \leqslant L||y_1 - y_2||$$
holds (i.e., h satisfies the Lipschitz condition with an arbitrarily small constant in a neighborhood of $y = 0$);

4) any solution $z(t) = (x(t), y(t))$ that starts on the surface h (i.e., there exists a moment of time $t = t_0$ such that $x(t_0) = h(t_0, y(t_0)))$ stays on the surface $x(t) = h(t, y(t))$ for $t \geqslant t_0$ and tends to the origin as $t \to +\infty$, satisfying the following estimate
$$||z(t)|| \leqslant ||z(t_0)|| \left(1 + \frac{b(k-1)}{2k}||z(t_0)||^{k-1}(t - t_0)\right)^{\frac{1}{1-k}}; \tag{7}$$

5) any solution that does not start on the surface h leaves the sector $||x|| \leqslant ||y||$ as t grows.

In the paper [7] it is proven that for system (1) there also exists a locally integral surface of "neutral" type defined by the equation
$$y = g(t, x), \tag{8}$$
where the function g satisfies the following conditions:

1) $g \in C(\mathbb{R} \times \{||x|| \leqslant a\} \mapsto \mathbb{R}^q)$;

2) $g(t, 0) = 0 \;\; \forall t \in \mathbb{R}$;

3)
$$||g(t, x_1) - g(t, x_2)|| \leqslant ||x_1 - x_2|| \text{ is true for } \forall ||x_1||, ||x_2|| \leqslant a \text{ and } \forall t \in \mathbb{R}; \tag{9}$$

4)
$$||g(t, x) - f(x)|| \leqslant \alpha/2||x||^{m/k}, \tag{10}$$

where $f(x)$ is the only solution to the equation $G(y) + F(x) = 0$ and $f(x)$ has the following properties:

1) $f \in C^1(\mathbb{R}^p)$;

2) f is a homogeneous function of degree m/k;

3) there exist constants $\alpha > 0, \beta > 0, T > 0$, such that
$$\alpha||x||^{m/k} \leqslant ||f(x)|| \leqslant \beta||x||^{m/k}, \tag{11}$$
$$||f'(x)|| \leqslant T||x||^{m/k-1}. \tag{12}$$

The existence of this solution is proven in [7].

Remark 1. If system (1) is ω-periodic with respect to t, then the functions h and g are also ω-periodic with respect to t. And if system (1) is autonomous, then h and g also do not depend on t.

Note that we are considering a system of ODEs of a somewhat special form due to the assumptions about the existence of the functions G and F. However, it was shown in the paper [7] that these assumptions are essential for the existence of a surface (8) of neutral type and the corresponding counterexample is given.

In this article, we prove the existence of an invariant foliation into locally integral surfaces of stable type for some neighborhood of the neutral surface g. This is proven under the additional assumption that the zero solution is uniformly stable on the neutral surface, i.e., the zero solution for the system

$$\dot{x} = X(t, x, g(t, x)), \tag{13}$$

is uniformly stable. Note that system (13) is a restriction of the original system (1) to the locally integral surface g.

Theorem 2. *Let the zero solution of system (13) be Lyapunov uniformly stable. Then the surface (8) has a neighbourhood $H(\delta_2)$, defined by formula (39) (see below), such that for any solution $z_\xi(t) = (x_\xi(t), y_\xi(t))$, that lies on the surface (8) and in the neighborhood $H(\delta_2)$ simultaneously, there exists only one locally integral surface of the form $x = h(t, y, \xi)$ passing through this solution. All the solutions on this surface tend to the solution $z_\xi(t)$ as t grows. The surfaces $x = h(t, y, \xi)$ fill the neighborhood $H(\delta_2)$ completely.*

Remark 2. If system (1) is periodic or autonomous, then the stability of the zero solution will be uniform automatically, as is well known. In this case, the uniformity is no longer required in the statement of the theorem.

The questions of the existence of integral surfaces and invariant foliations are one of the most important in the qualitative theory of differential equations. These questions have been most fully studied for quasilinear systems, i.e., systems with a linear approximation that is hyperbolic in some sense. On the contrary, we still know very little about essentially nonlinear systems. When studying such systems, we are faced primarily with two difficulties. First, we have to look for those conditions which will ensure the existence of the required integral surfaces. It is desirable for the applications that these conditions have a form testable via coefficients. In this article, this role is played by logarithmic norms. Secondly, we have to look for the proof methods that will work effectively with essentially nonlinear systems. We will use the so-called graph transformation method, which goes back to Hadamard. The proof in the present article will be based on the scheme of application of this method, which was proposed by Pliss in [10, 11]. Note that since this method only works easy for the cases of quasilinear, periodic, and autonomous systems, we will be forced to substantially adapt it in this article.

4. PROOF OF THEOREM 2

To prove the theorem we first prove that for each solution $z_\xi(t)$ on the surface (8) there is a unique "stable" locally integral surface of the required form passing through $z_\xi(t)$ (Lemma 1). After that we show that these surfaces fill the neighborhood $H(\delta_2)$ from the statement of Theorem 2 completely (Lemma 9). The proof of Lemma 1 is quite long, and therefore we split it into Lemmas 2–8.

Denote by $z(t, t_0, \xi)$ the solution to system (1) that lies on the surface (8) $y = g(t, x)$ and passes through the point $t = t_0$, $z = (\xi, g(t_0, \xi))$. By virtue of the uniform stability assumption, there exists a $\triangle > 0$ independent of t_0 such that for any $||\xi|| < \triangle$, the solution $z(t, t_0, \xi)$ is extendable to $[t_0, +\infty)$ and satisfies the estimate $||z(t, t_0, \xi)|| \leqslant 1$. This estimate is introduced to simplify some of the further calculations. Therefore, we assume that the condition $||x|| < \triangle$ is satisfied everywhere. This allows, in particular, to consider $a \leqslant 1$ without any loss of generality.

Let us define the set

$$H(t_0, \xi) \stackrel{\text{def}}{=} \{(t, x, y) : \; ||x - x(t, t_0, \xi)|| \leqslant ||y - y(t, t_0, \xi)|| \leqslant \frac{\alpha}{4} ||x(t, t_0, \xi)||^{\frac{m}{k}},$$

$$||y|| \leqslant ||x|| \leqslant a, t \geqslant t_0\}. \tag{14}$$

It follows from (10) and (11) that for the solution $z(t,t_0,\xi)$ located on the surface (8), the following inequalities hold:

$$\frac{\alpha}{2}||x(t,t_0,\xi)||^{\frac{m}{k}} \leqslant ||y(t,t_0,\xi)|| \leqslant \left(\frac{\alpha}{2}+\beta\right)||x(t,t_0,\xi)||^{\frac{m}{k}}. \tag{15}$$

Taking into account that $||x(t,t_0,\xi)|| \leqslant 1$, we see that for any $(t,x,y) \in H(t_0,\xi)$

$$\frac{\alpha}{4}||x(t,t_0,\xi)||^{\frac{m}{k}} \leqslant ||y|| \leqslant \left(\frac{3\alpha}{4}+\beta\right)||x(t,t_0,\xi)||^{\frac{m}{k}}, \tag{16}$$

$$||y|| \leqslant ||x|| \leqslant \left(1+\frac{\alpha}{4}\right)||x(t,t_0,\xi)||. \tag{17}$$

Lemma 1. *For sufficiently small $||\xi||$, for each solution $z(t,t_0,\xi)$ there exists a unique locally integral surface of system (1), that is located in the set $H(t_0,\xi)$ and passes through $z(t,t_0,\xi)$. This surface is represented by the formula*

$$x - x(t,t_0,\xi) = h_\xi(y - y(t,t_0,\xi), t, t_0), \tag{18}$$

where the function $h_\xi : \{(y,t,t_0):\ ||y|| \leqslant \frac{\alpha}{4}||x(t,t_0,\xi)||^{\frac{m}{k}}, t \geqslant t_0\} \mapsto \mathbb{R}^p$ is continuous with respect to y and t, $h(0,t,t_0) = 0$, and for any $L \in (0;1]$ one can specify $\delta(L) > 0$ such that if $||\xi|| \leqslant \delta(L)$, then

$$||h_\xi(y_1,t,t_0) - h_\xi(y_2,t,t_0)|| \leqslant L||y_1 - y_2||. \tag{19}$$

Any solution $z(t)$ of system (1) that lies on (18) satisfies the inequalities

$$||z(t) - z(t,t_0,\xi)|| \leqslant \frac{\alpha}{4}||x(t,t_0,\xi)||^{\frac{m}{k}}, \tag{20}$$

$$||z(t) - z(t,t_0,\xi)|| \leqslant ||z(t_0) - z(t_0,t_0,\xi)|| \exp\left(-\nu\int_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds\right), \tag{21}$$

where the constant ν is defined by formula (34) (see below) and $t \geqslant t_0$.

Proof of Lemma 1. Let us make the change of variables in system (1): $u = x - x(t,t_0,\xi), v = y - y(t,t_0,\xi)$. As a result, we get the system

$$\dot{u} = U(t,w), \quad \dot{v} = A(t,v) + B(t,u) + V(t,w), \tag{22}$$

where $w = (u,v), U(t,w) = X(t,w+z(t,t_0,\xi)) - X(t,z(t,t_0,\xi)),\quad A(t,w) = G(t,v+y(t,t_0,\xi)) - G(t,y(t,t_0,\xi)),\quad B(t,u) = G(t,u+x(t,t_0,\xi)) - G(t,x(t,t_0,\xi)),\quad V(t,w) = Y(t,w+z(t,t_0,\xi)) - Y(t,z(t,t_0,\xi))$. With respect to the new variables u and v the set $H(t_0,\xi)$ takes the following form:

$$\widetilde{H}(t_0,\xi) = \{(t,u,v):\ ||u|| \leqslant ||v|| \leqslant \frac{\alpha}{4}||x(t,t_0,\xi)||^{\frac{m}{k}},$$

$$||v + y(t,t_0,\xi)|| \leqslant ||u + x(t,t_0,\xi)|| \leqslant a, t \geqslant t_0\}. \tag{23}$$

To prove Lemma 1, it is sufficient to show that system (22) has a locally integral surface that can be represented by the formula $u = h_\xi(v,t,t_0)$, lies in $\widetilde{H}(t_0,\xi)$, and satisfies the required properties. This proof is split into a sequence of Lemmas 2–8. We start by stating some estimates for A, B, U, and V on the set $\widetilde{H}(t_0,\xi)$. Denote $\Delta w = w_1 - w_2 = (\Delta u, \Delta v)$. We consider $||\Delta u|| \leqslant ||\Delta v||$. Then, according to our definition of $||z||$ (formula (6)), $||\Delta w|| = ||\Delta v||$. Also, the equality $||w + z(t,t_0,\xi)|| = ||u + x(t,t_0,\xi)||$ follows from the definition of $\widetilde{H}(t_0,\xi)$. Taking (17) into consideration, we get

$$||w + z(t,t_0,\xi)|| \leqslant \left(1+\frac{\alpha}{4}\right)||x(t,t_0,\xi)||. \tag{24}$$

The following auxiliary inequality is also easily verified:

$$d\max\{||y_1||^{k-1}, ||y_2||^{k-1}\} \leqslant \int_0^1 ||y_2 + \theta\Delta y||^{k-1}d\theta \leqslant \max\{||y_1||^{k-1}, ||y_2||^{k-1}\}, \tag{25}$$

where $d \overset{\text{def}}{=} \min\left\{\int_0^1 ||s_2 + \theta\Delta(s_1 - s_2)||^{k-1} d\theta : \ ||s_1|| \leqslant 1, ||s_1|| = 1\right\} > 0$.

Using the Mean Value Theorem, we get

$$||U(t, w_1) - U(t, w_2)|| \leqslant \int_0^1 ||X'_z(t, w_1 + z(t, t_0, \xi) + \theta\Delta w)|| d\theta \, ||\Delta w||$$

$$\overset{(25),(4)}{\leqslant} C_1 \max_{i=1,2} ||w_i + z(t, t_0, \xi)||^{m-1} ||\Delta v|| \overset{(24)}{\leqslant} C_1 \left(1 + \frac{\alpha}{4}\right)^{m-1} ||x(t, t_0, \xi)||^{m-1} ||\Delta v||. \tag{26}$$

Likewise,

$$||V(t, w_1) - V(t, w_2)|| \leqslant C_1 \left(1 + \frac{\alpha}{4}\right)^{m-1} ||x(t, t_0, \xi)||^{m-1} ||\Delta v||. \tag{27}$$

For $B(t, u)$ we get

$$||B(t, u_1) - B(t, u_2)|| \leqslant \int_0^1 ||F'_x(u_2 + x(t, t_0, \xi) + \theta\Delta u)|| d\theta \, ||\Delta u||$$

$$\overset{(25),(3)}{\leqslant} r_2 \max_{i=1,2} ||u_i + x(t, t_0, \xi)||^{m-1} ||\Delta u|| \overset{(24)}{\leqslant} r_2 \left(1 + \frac{\alpha}{4}\right)^{m-1} ||x(t, t_0, \xi)||^{m-1} ||\Delta v||. \tag{28}$$

As for $A(t, v)$, let us consider the system

$$\frac{d\Delta v}{dt} = A(t, v_1) - A(t, v_2) = \int_0^1 A'_v(t, v_2 + \theta\Delta v) d\theta \, \Delta v.$$

Applying Theorem 1 and property 4 of the logarithmic norm, as well as formulas (2), (25) and (16), we obtain

$$\frac{d_+ ||\Delta v||}{dt} \leqslant \int_0^1 \gamma^*(G'(v_2 + y(t, t_0, \xi) + \theta\Delta v)) d\theta \, ||\Delta v||$$

$$\overset{(2)}{\leqslant} -b \int_0^1 ||v_2 + y(t, t_0, \xi) + \theta\Delta v||^{k-1} d\theta \, ||\Delta v|| \overset{(25)}{\leqslant} -bd \max_{i=1,2} ||v_i + y(t, t_0, \xi)||^{k-1} ||\Delta v||$$

$$\overset{(16)}{\leqslant} -bd \left(\frac{\alpha}{4}\right)^{k-1} ||x(t, t_0, \xi)||^{m-\frac{m}{k}} ||\Delta v||. \tag{29}$$

Lemma 2. *There exists a $\delta_1 > 0$ independent of t_0 such that if $||\xi|| \leqslant \delta_1$, then the surface $\Pi : ||u|| = ||v|| > 0$ is a set of strict egress points from $\widetilde{H}(t_0, \xi)$ for solutions of system (22) (i.e., strict ingress points if t is decreasing) (see Fig. 1).*

Proof of Lemma 2. Concerning egress or ingress points, see [9]. Consider the function $\Phi(w) = ||u|| - ||v||$. It is obvious that $\Phi|_{\widetilde{H}(t_0,\xi)} < 0$, $\Phi|_\Pi = 0$. Therefore, it is sufficient to show that $D_+\Phi|_\Pi > 0$, where D_+ denotes the right-hand derivative due to system (22). Applying Theorem 1 and estimates (26)–(29) with $w_1 = w, w_2 = 0$, we get

$$D_+\Phi|_\Pi = D_+||u|| - D_+||v|| \geqslant -||U(t, w)||$$

$$+ bd \left(\frac{\alpha}{4}\right)^{k-1} ||x(t, t_0, \xi)||^{m-\frac{m}{k}} ||v|| - ||B(t, u)|| - ||V(t, w)||$$

$$\geqslant \left[bd \left(\frac{\alpha}{4}\right)^{k-1} - (2C_1 + r_2)\left(1 + \frac{\alpha}{4}\right)^{m-1} ||x(t, t_0, \xi)||^{\frac{m}{k}-1}\right] ||x(t, t_0, \xi)||^{m-\frac{m}{k}} ||v||.$$

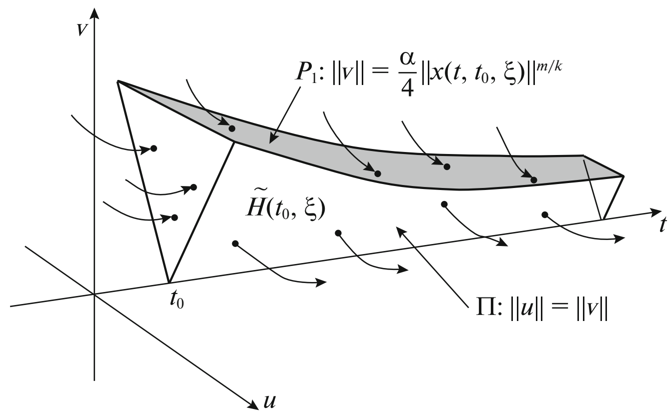

Fig. 1.

Since the zero solution is uniformly stable on the surface (8), there exists a $\delta_1 > 0$ independent of t_0, such that if $||\xi|| \leqslant \delta_1$, then the inequality $||x(t,t_0,\xi)|| \leqslant \varepsilon_1$ holds for $t \geqslant t_0$. Here ε_1 is defined by the following formula:

$$\varepsilon_1 \overset{\text{def}}{=} \left(0.5\, bd\left(\frac{\alpha}{4}\right)^{k-1}(2C_1+r_2)^{-1}\left(1+\frac{\alpha}{4}\right)^{1-m}\right)^{\frac{k}{m-k}}. \tag{30}$$

With this choice of ε_1, the expression in square brackets will be $\geqslant 0.5\left(bd\left(\frac{\alpha}{4}\right)^{k-1}\right)$. Then, for $||\xi|| \leqslant \delta_1$, we have $D_+\Phi|_\Pi > 0$, and thus Lemma 2 is proven. ■

Consider a pair of solutions to system (22): $w_i(t) = (u_i(t), v_i(t)), i = 1, 2$.

Lemma 3. *For any $L \in (0;1]$, a $\delta(L) > 0$ can be specified so that for all $||\xi|| < \delta(L)$ the following statement is true: if $(\tau, w_i(\tau)) \in \widetilde{H}(t_0,\xi), i = 1,2$ for all $\tau \in [t_0,t]$ and $||\Delta u(t)|| \leqslant L||\Delta v(t)||$ holds, then $||\Delta u(\tau)|| \leqslant L||\Delta v(\tau)$ for all $\tau \in [t_0,t]$.*

Proof of Lemma 3. To prove this lemma, it is sufficient to show that if the equation $||\Delta u(\tau^*)|| = L||\Delta v(\tau^*)||$ holds true for some $\tau^* \leqslant t$, then

$$\frac{d_-||\Delta u(\tau)||}{d\tau} - L\,\frac{d_-||\Delta v(\tau)||}{d\tau}\bigg|_{\tau=\tau^*} < 0.$$

Since $||\Delta u(\tau^*)|| \geqslant ||\Delta v(\tau^*)||$, from definition (6) and norm $||w||$ we get $||\Delta w(\tau^*)|| = ||\Delta u(\tau^*)||$. Applying Theorem 1 and estimates (26)–(29), we see that

$$\frac{d_-||\Delta u(\tau)||}{d\tau} - L\,\frac{d_-||\Delta v(\tau)||}{d\tau}\bigg|_{\tau=\tau^*} \leqslant ||U(\tau^*, w_1(\tau^*)) - U(\tau^*, w_2(\tau^*))||$$

$$+ L\Bigg(\gamma^*\left(\int\limits_0^1 A_v'(\tau^*, v_2(\tau^*) + \theta\Delta v(\tau^*))\,d\theta\right)||\Delta v(\tau^*)||$$

$$+ ||B(\tau^*, u_1(\tau^*)) - B(\tau^*, u_2(\tau^*))|| + ||V(\tau^*, w_1(\tau^*)) - V(\tau^*, w_2(\tau^*))||\Bigg)$$

$$\leqslant -Lbd\left(\frac{\alpha}{4}\right)^{k-1}||x(\tau^*, t_0, \xi)||^{m-\frac{m}{k}}||\Delta v(\tau^*)||$$

$$+ \big((1+L)C_1 + Lr_2\big)\left(1+\frac{\alpha}{4}\right)^{m-1}||x(\tau^*, t_0, \xi)||^{m-1}||\Delta v(\tau^*)||$$

$$= \left[-Lbd\left(\frac{\alpha}{4}\right)^{k-1} + \big((1+L)C_1 + Lr_2\big)\left(1+\frac{\alpha}{4}\right)^{m-1}||x(\tau^*, t_0, \xi)||^{\frac{m}{k}-1}\right]$$

$$\times ||x(\tau^*, t_0, \xi)||^{m-\frac{m}{k}}||\Delta v(\tau^*)||.$$

According to the uniform stability assumption, such a $\delta(L) > 0$ independent of t_0 can be found so that for $||\xi|| \leqslant \delta(L)$ and $t \geqslant t_0$ the inequality $||x(t, t_0, \xi)|| \leqslant \varepsilon_L$ holds, where ε_L is chosen according to the condition

$$-Lbd\left(\frac{\alpha}{4}\right)^{k-1} + \left((1+L)C_1 + Lr_2\right)\left(1+\frac{\alpha}{4}\right)^{m-1} {\varepsilon_L}^{\frac{m}{k}-1} < 0.$$

Therefore, for $||\xi|| \leqslant \delta(L)$ the expression in square brackets will always be negative, and so $\frac{d_-||\Delta u(\tau)||}{d\tau} - L\frac{d_-||\Delta v(\tau)||}{d\tau}\Big|_{\tau=\tau^*} < 0$. Thus, Lemma 3 is proven. ■

Lemma 4. *The norm $||v||$ decreases along solutions of system (22) in $\widetilde{H}(t_0, \xi)$ for $||\xi|| \leqslant \delta_1$, where δ_1 is taken from Lemma 2.*

Proof of Lemma 4. Let us use Theorem 1 and estimates (27)–(29) again. We have

$$D_+||v||\Big|_{\widetilde{H}} \leqslant \gamma^*\left(\int\limits_0^1 A'_v d\theta\right)||v|| + ||B(t,u)|| + ||V(t,w)||$$

$$\leqslant \left(-bd\left(\frac{\alpha}{4}\right)^{k-1} + \left(C_1 + r_2\right)\left(1+\frac{\alpha}{4}\right)^{m-1}||x(t,t_0,\xi)||^{\frac{m}{k}-1}\right)||x(t,t_0,\xi)||^{m-\frac{m}{k}}||v||.$$

According to the definition of δ_1, from the condition $||\xi|| \leqslant \delta_1$ it follows that $||x(t, t_0, \xi)|| \leqslant \varepsilon_1$ for all $t \geqslant t_0$, where ε_1 satisfies (30). Then

$$D_+||v||\Big|_{\widetilde{H}} \leqslant \left(-bd\left(\frac{\alpha}{4}\right)^{k-1} + \left(C_1 + r_2\right)\left(1+\frac{\alpha}{4}\right)^{m-1}\varepsilon_1^{\frac{m}{k}-1}\right)||x(t,t_0,\xi)||^{m-\frac{m}{k}}||v||$$

$$\leqslant -0.5\,bd\left(\frac{\alpha}{4}\right)^{k-1}||x(t,t_0,\xi)||^{m-\frac{m}{k}}||v||. \tag{31}$$

This inequality proves Lemma 4. ■

Lemma 5. *If the solution $w(t) = (u(t), v(t))$ to system (22) is such that $(t, w(t)) \in \widetilde{H}(t_0, \xi)$ for $t \in [t_0, t_1]$ and $||\xi|| \leqslant \delta_1$, then for $t \in [t_0, t_1]$ this solution satisfies the following estimates:*

$$||w(t)|| \leqslant \frac{\alpha}{4}||x(t,t_0,\xi)||^{\frac{m}{k}}, \tag{32}$$

$$||w(t)|| \leqslant ||w(t_0)||\exp\left(-\nu\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds\right), \tag{33}$$

where

$$\nu \overset{\text{def}}{=} 0.5\,bd\left(\frac{\alpha}{4}\right)^{k-1}. \tag{34}$$

Proof of Lemma 5. Estimate (32) follows directly from the definition (23) of the set $\widetilde{H}(t_0, \xi)$. As for estimate (33), it can be obtained by integrating inequality (31), taking the equation $||w|| = ||v||$ into account. Lemma 5 is proven. ■

Lemma 6. *Let $u_1 \neq u_2$ and $(t_0, u_i, v_0) \in \widetilde{H}(t_0, \xi)$, where $i = 1, 2$. Then at least one of the solutions $w_1(t) = w(t, t_0, u_1, v_0)$ or $w_2(t) = w(t, t_0, u_2, v_0)$ of system (22) leaves $\widetilde{H}(t_0, \xi)$ as t grows.*

Proof of Lemma 6. Let us assume the opposite, i.e.. let both solutions $w_1(t)$ and $w_2(t)$ stay in $\widetilde{H}(t_0, \xi)$ for all $t \geqslant t_0$. Denote $w_i(t) = (u_i(t), v_i(t))$ and $\Delta w(t) = (\Delta u(t), \Delta v(t)) = w_1(t) - w_2(t)$. Since $0 = ||v_0 - v_0|| = ||\Delta v(t_0)|| < ||\Delta u(t_0)|| = ||u_1 - u_2|| \neq 0$, using Lemma 3 we get $||\Delta v(t)|| < ||\Delta u(t)||$ for all $t \geqslant t_0$. Hence, $||\Delta w(t)|| = ||\Delta u(t)||$. The equation for $\Delta u(t)$ takes the following form:

$$\frac{d\Delta u(t)}{dt} = \int\limits_0^1 U'_w(t, w_2(t) + \theta\Delta w(t))\,d\theta\ \Delta w(t).$$

After using (26) we have

$$\frac{d_+||\Delta u(t)||}{dt} \geqslant -C_1\left(1+\frac{\alpha}{4}\right)^{m-1}||x(t,t_0,\xi)||^{m-1}||\Delta u(t)||.$$

By integrating this inequality from t_0 to t, we get

$$\ln\frac{||\Delta u(t)||}{||\Delta u(t_0)||} \geqslant -C_1\left(1+\frac{\alpha}{4}\right)^{m-1}\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-1}ds. \tag{35}$$

On the other side, applying (33), we can write

$$||\Delta u(t)|| \leqslant \sum_1^2||u_i(t)|| \leqslant \sum_1^2||w_i(t)|| \overset{(33)}{\leqslant} \sum_1^2||w_i(t_0)||\exp\left(-\nu\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds\right)$$

$$= \sum_1^2||v_i(t_0)||\exp\left(-\nu\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds\right)$$

(we use the fact that the equality $||w(t)|| = ||v(t)||$ holds in $\widetilde{H}(t_0,\xi)$). Let us take the logarithm of this inequality:

$$\ln||\Delta u(t)|| \leqslant \ln\sum_1^2||v_i(t_0)|| - \nu\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds.$$

Combining it with the previously written inequality (35), we obtain

$$\ln\sum_1^2||v_i(t_0)|| - \nu\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds \geqslant \ln||\Delta u(t_0)|| - C_1\left(1+\frac{\alpha}{4}\right)^{m-1}\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-1}ds.$$

This inequality can be rewritten in the following form:

$$\int\limits_{t_0}^{t}\left(\nu - C_1\left(1+\frac{\alpha}{4}\right)^{m-1}||x(s,t_0,\xi)||^{\frac{m}{k}-1}\right)||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds \leqslant C^*,$$

where C^* is some positive constant. Here, the exact value of this constant is not important.

According to the choice of δ_1 we made in the Lemma 2, for $||\xi|| \leqslant \delta_1$ and $t \geqslant t_0$ the inequality $||x(t,t_0,\xi)|| \leqslant \varepsilon_1$ holds, where ε_1 satisfies (30). Taking this in consideration alongside the definition (34) of the constant ν, we have

$$\nu - C_1\left(1+\frac{\alpha}{4}\right)^{m-1}||x(s,t_0,\xi)||^{\frac{m}{k}-1} \geqslant \nu - C_1\left(1+\frac{\alpha}{4}\right)^{m-1}\varepsilon_1^{\frac{m}{k}-1}$$

$$\geqslant 0.5\,bd\left(\frac{\alpha}{4}\right)^{k-1} - 0.5\,C_1\left(1+\frac{\alpha}{4}\right)^{m-1}bd\left(\frac{\alpha}{4}\right)^{k-1}\frac{1}{r_2+2C_1}\left(1+\frac{\alpha}{4}\right)^{1-m}$$

$$\geqslant (0.5-0.25)bd\left(\frac{\alpha}{4}\right)^{k-1} = 0.25\,\nu > 0.$$

Hence,

$$0.25\,\nu\int\limits_{t_0}^{t}||x(s,t_0,\xi)||^{m-\frac{m}{k}}ds \leqslant C^*.$$

In the paper [7] it is shown that the solution $||z(t,t_0,\xi)||$ that lies on the surface (8), satisfies the following estimate for $t \geqslant t_0$:

$$||z(t,t_0,\xi)|| = ||x(t,t_0,\xi)|| \geqslant ||\xi|| \left(1 + \frac{C_1(m-1)}{m}||\xi||^{m-1}(t-t_0)\right)^{-\frac{1}{m-1}}.$$

This means that even if the solution lying on the neutral surface (8) tends to zero as t grows, then the speed of the decrease is estimated from below by some negative power of $(t-t_0)$. Using this estimate for $||x(s,t_0,\xi)||$ under the integral sign, we finally obtain that, assuming the opposite of Lemma 6, for all $t \geqslant t_0$ we will have

$$C^* \geqslant 0.25\,\nu\,||\xi||^{m-\frac{m}{k}} \int\limits_{t_0}^{t} \left(1 + \frac{C_1(m-1)}{m}||\xi||^{m-1}(s-t_0)\right)^{-\frac{1}{m-1}(m-\frac{m}{k})} ds.$$

However, $\frac{1}{m-1}(m-\frac{m}{k}) < 1$. Hence, the integral on the right-hand side diverges as $t \to +\infty$ and therefore cannot be bounded. This contradiction proves that at least one of the solutions $w_1(t)$ or $w_2(t)$ must leave $\widetilde{H}(t_0,\xi)$ as t grows. Thus, Lemma 6 is proven. ■

Lemma 7. *For $||\xi|| \leqslant \delta_1$, the surface $P_1:\ ||v|| = \frac{\alpha}{4}||x(t,t_0,\xi)||^{m/k}$ is a set of strict ingress points to the $\widetilde{H}(t_0,\xi)$ (see Fig. 1).*

Proof of Lemma 7. Similarly to the proof of Lemma 2, we define the function $\Phi(w) = ||v|| - \frac{\alpha}{4}||x(t,t_0,\xi)||^{m/k}$, that vanishes on P_1, and show that $D_+\Phi\big|_{P_1} < 0$.

It follows from the inequality $D_+||x|| \leqslant ||X(t,z)|| \leqslant \frac{C_1}{m}||z||^m$ that $D_+||x(t,t_0,\xi)|| \leqslant \frac{C_1}{m}||x(t,t_0,\xi)||^m$ since $||z(t,t_0,\xi)|| = ||x(t,t_0,\xi)||$. For $D_+||v||$, we use the estimate obtained in Lemma 4. Therefore,

$$\begin{aligned} D_+\Phi\big|_{P_1} &\leqslant D_+||v|| - \frac{\alpha m}{4k}\,||x(t,t_0,\xi)||^{m/k-1} D_+||x(t,t_0,\xi)|| \\ &\leqslant \left(-bd\left(\frac{\alpha}{4}\right)^{k-1} + (C_1+r_2)\left(1+\frac{\alpha}{4}\right)^{m-1}||x(t,t_0,\xi)||^{\frac{m}{k}-1}\right)||x(t,t_0,\xi)||^{m-\frac{m}{k}}||v|| \\ &\quad + \frac{\alpha m}{4k}\,||x(t,t_0,\xi)||^{m/k-1}\frac{C_1}{m}\,||x(t,t_0,\xi)||^m \end{aligned}$$

(we take into account that $||v|| = \frac{\alpha}{4}\,||x(t,t_0,\xi)||^{m/k}$)

$$\begin{aligned} &= \left(-bd\left(\frac{\alpha}{4}\right)^{k-1} + (C_1+r_2)\left(1+\frac{\alpha}{4}\right)^{m-1}||x(t,t_0,\xi)||^{\frac{m}{k}-1}\right)\frac{\alpha}{4}\,||x(t,t_0,\xi)||^m \\ &\quad + \frac{\alpha m}{4k}\,||x(t,t_0,\xi)||^{m/k-1}\frac{C_1}{m}\,||x(t,t_0,\xi)||^m \\ &= \left[-bd\left(\frac{\alpha}{4}\right)^{k-1} + \left((C_1+r_2)\left(1+\frac{\alpha}{4}\right)^{m-1} + \frac{C_1}{k}\right)||x(t,t_0,\xi)||^{\frac{m}{k}-1}\right]\frac{\alpha}{4}\,||x(t,t_0,\xi)||^m. \end{aligned}$$

Comparing this inequality with the one obtained in the proof of Lemma 2, we see that

$$\begin{aligned} &\left[-bd\left(\frac{\alpha}{4}\right)^{k-1} + \left((C_1+r_2)\left(1+\frac{\alpha}{4}\right)^{m-1} + \frac{C_1}{k}\right)||x(t,t_0,\xi)||^{\frac{m}{k}-1}\right] \\ &\leqslant \left[-bd\left(\frac{\alpha}{4}\right)^{k-1} + (2C_1+r_2)\left(1+\frac{\alpha}{4}\right)^{m-1}||x(t,t_0,\xi)||^{\frac{m}{k}-1}\right]. \end{aligned}$$

It was shown in the proof of Lemma 2 that for $||\xi|| \leqslant \delta_1$, the second square bracket will be less than $-0.5\left(bd\left(\frac{\alpha}{4}\right)^{k-1}\right)$. Therefore, $D_+\Phi\big|_{P_1} < 0$, and thus Lemma 7 is proven. ■

Define

$$\bar{\delta}(L) \stackrel{\text{def}}{=} \min\{\delta_1, \delta(L)\}, \tag{36}$$

where δ_1 is taken from Lemma 1 and $\delta(L)$ is taken from Lemma 3. Let us fix an arbitrary $L \in (0;1]$ and an arbitrary solution $z(t,t_0,\xi)$ with $||\xi|| \leqslant \bar{\delta}(L)$. Define the space $K(L,\xi,t,t_0)$ of continuous functions $u = h(v)$ with the following properties:

$$h:\ \{v:\ ||v|| \leqslant \frac{\alpha}{4}\,||x(t,t_0,\xi)||^{m/k}, t \geqslant t_0\} \mapsto \mathbb{R}^p,$$

$$h(0) = 0, \quad ||h(v_1) - h(v_2)|| \leqslant L||v_1 - v_2||.$$

We define a metric on $K(L,\xi,t,t_0)$ by formula $\rho(h_1,h_2) = \max ||h_1(v) - h_2(v)||$, where max is taken over all v from the domain, which is compact. It is easy to see that the space (K,ρ) is complete (i.e., is a Cauchy space).

Let us introduce an operator $\mathcal{F}_{t_1,t_2}$, where $t_1 \geqslant t_2 \geqslant t_0$, that assigns to each function $h \in K(L,\xi,t,t_0)$ the set

$$\{(u,v):\ (u,v) = w(t_2,t_1,h(p),p)\}, \tag{37}$$

where we assume that p runs over the entire domain of h and $(\tau, w(\tau,t_1,h(p),p)) \in \widetilde{H}(t_0,\xi)\ \forall \tau \in [t_2,t_1]$. The operator $\mathcal{F}_{t_1,t_2}$ is a shift map from time t_1 to time t_2 along the solutions of system (22). Note that we are considering a reverse shift, since $t_2 \leqslant t_1$.

Let us show that the operator $\mathcal{F}_{t_1,t_2}$ acts from $K(L,\xi,t_1,t_0)$ to $K(L,\xi,t_2,t_0)$ in the sense that set (37) is the graph of some function from $K(L,\xi,t_2,t_0)$. Indeed, it follows from Lemma 3 and the general theorem on the continuous dependence of solutions on the initial data that (37) is the graph of some function satisfying the Lipschitz condition with constant L. The fact that the domain of this function is the whole ball $\{v:\ ||v|| \leqslant \frac{\alpha}{4}||x(t_2,t_0,\xi)||^{m/k}\}$ follows from Lemma 7.

Let us consider an arbitrary function

$$h:\ \{v:\ ||v|| \leqslant \frac{\alpha}{4}\ (\varepsilon_L)^{\frac{m}{k}}, t \geqslant t_0\} \mapsto \mathbb{R}^p,$$

such that $h(0) = 0$, $||h(v_1) - h(v_2)|| \leqslant L||v_1 - v_2||$ (ε_L here is taken from Lemma 3). It is easy to see that the restriction of h on the set $\{v:\ ||v|| \leqslant \frac{\alpha}{4}||x(t,t_0,\xi)||^{m/k}\}$ belongs to $K(L,\xi,t,t_0)$ for any $t \geqslant t_0$ and $||\xi|| \leqslant \bar{\delta}(L)$. Denote the function whose graph is given by formula (37) by $\mathcal{F}_{t_1,t_2}h$.

Lemma 8. *For any $t \geqslant t_0$, there exists the limit of the sequence $\{\mathcal{F}_{n,t}h\}$ as $n \to +\infty$ in the metric of the space $(K(L,\xi,t,t_0),\rho)$.*

Proof of Lemma 8. Let us notice that due to the Lipschitz condition, all the functions from $K(L,\xi,t,t_0)$ are uniformly bounded and equicontinuous. Therefore, by the Arzela–Ascoli lemma, the set $(K(L,\xi,t,t_0),\rho)$ is relatively compact. Thus, from the sequence $\{\mathcal{F}_{n,t}h\}$ a converging subsequence $\{\mathcal{F}_{n_m,t}h\}$ can be selected. Let us denote $h^t = \lim\limits_{m\to+\infty} \mathcal{F}_{n_m,t}h \in K(L,\xi,t,t_0)$. Note that the solution $w(\tau,t,h^t(v_0),v_0)$ of system (22) stays in $\widetilde{H}(t_0,\xi)$ for all $\tau \geqslant t$. Indeed, if this is not the case, then there exists a moment of time $\tau_1 > t$ such that $(\tau_1, w(\tau_1,t,h^t(v_0),v_0)) \notin \widetilde{H}(t_0,\xi)$. Since $\mathcal{F}_{n_m,t}h \xrightarrow{\rho} h^t$ for $m \to +\infty$ and the set $\widetilde{H}(t_0,\xi)$ is closed, then by the theorem on the continuous dependence of solutions on the initial data there exists a natural M such that for all $m > M$,

$$(\tau_1, w(\tau_1,t,\mathcal{F}_{n_m,t}h(v_0),v_0)) \notin \widetilde{H}(t_0,\xi). \tag{38}$$

However, by the definition of $\mathcal{F}_{n_m,t}h$, the solution $w(\tau,t,\mathcal{F}_{n_m,t}h(v_0),v_0)$ belongs to $\widetilde{H}(t_0,\xi)$ at least for $\tau \in [t,n_m]$. Since $n_m \to +\infty$, then for sufficiently large m, $\tau_1 \in [t,n_m]$ will hold true. Therefore (38) is impossible, and we have arrived at a contradiction. Thus, $(\tau, w(\tau,t,h^t(v_0),v_0)) \in \widetilde{H}(t_0,\xi)$ for all $\tau \geqslant t$.

Now let us consider the opposite to the statement of Lemma 8. Then from the sequence $\{\mathcal{F}_{n,t}h\}$ we can select another subsequence converging to some function $\widetilde{h}^t \in K(L,\xi,t,t_0)$ different from h^t. Let v_0 be such that $\widetilde{h}^t(v_0) \neq h^t(v_0)$. Then at least one of the solutions $w(\tau_1,t,h^t(v_0),v_0)$ or $w(\tau_1,t,\widetilde{h}^t(v_0),v_0)$ should leave $\widetilde{H}(t_0,\xi)$ as τ grows according to Lemma 6. But this contradicts the fact that both of them, as was shown above, must remain in $\widetilde{H}(t_0,\xi)$. This contradiction proves Lemma 8. ■

Let us define a vector-function $h_\xi(v,t,t_0)$ that represents the required locally integral surface by the relation

$$h_\xi(v,t,t_0) \stackrel{\text{def}}{=} \lim_{n\to+\infty} \mathcal{F}_{n,t}h(v).$$

It follows from the equality $\mathcal{F}_{t_1,t_2}h_\xi(v,t_1,t_0) = h_\xi(v,t_2,t_0)$ that the surface $u = h_\xi(v,t,t_0)$ is indeed locally integral, i.e., it consists of arcs of integral curves. Moreover, this equality and the theorem on the continuous dependence of solutions on the initial data imply the continuity of h_ξ with respect to t. Since h_ξ satisfies the Lipschitz condition with respect to v uniformly with respect to t, then h_ξ will be continuous with respect to the set of variables (v,t).

Any solution starting on the surface $u = h_\xi(v,t,t_0)$ remains in $\widetilde{H}(t_0,\xi)$ for all $t \geqslant t_0$ and, according to Lemma 5, satisfies estimates (20) and (21). The uniqueness of the found surface follows from Lemma 6. Formula (19) follows from Lemma 3, where $\delta(L)$ is defined. As a result, Lemma 1 is completely proved. ■

Let us move on to the proof of the fact that the found surfaces completely fill some neighborhood of the neutral surface (8) $y = g(t,x)$. Note that the uniqueness of h_ξ implies that for fixed t and t_0, the function h_ξ continuously depends on ξ in the metric ρ introduced before Lemma 8, i.e., $h_\xi \xrightarrow{\rho} h_{\xi^*}$ as $\xi \to \xi^*$ for all $||\xi^*|| \leqslant \bar{\delta}(L)$. Let L_ξ denote the smallest Lipschitz constant for h_ξ. It follows from Lemma 1 that $L_\xi \to 0$ for $||\xi|| \to 0$.

Let us define the following set:

$$H(\delta) = \bigcup_{t_0\in\mathbb{R},\ ||\xi||\leqslant\min\{\bar{\delta}(L),\delta\}} H(t_0,\xi). \tag{39}$$

Note that $\bar{\delta}(L)$ defined by the formula (36) does not depend on t_0. The set $H(\delta)$ is some neighborhood of the surface (8). Proving the following lemma is sufficient to complete the proof of Theorem 2.

Lemma 9. *There exists a $\delta_2 > 0$ such that for any solution $z(t) = z(t,t_0,z_0)$ of system (1) with initial data $(t_0,z_0) \in H(\delta_2)$ there exists a $\xi :\ ||\xi|| \leqslant \delta_2$ such that the solution $z(t)$ is located on some surface of the form (18) passing through $z(t,t_0,\xi)$, i.e.,*

$$x(t) - x(t,t_0,\xi) = h_\xi(y(t) - y(t,t_0,\xi),t,t_0). \tag{40}$$

Proof of Lemma 9. Since the surface h_ξ is integral, equality (40) holds for all $t \geqslant t_0$ if it holds for $t = t_0$:

$$x_0 - \xi = h_\xi(y_0 - g(t_0,\xi),t_0,t_0), \tag{41}$$

where $z_0 = (x_0,y_0)$, $z(t_0,t_0,\xi) = (\xi,g(t_0,\xi))$. Let us consider (41) as the equations for finding the unknown $\xi \in \mathbb{R}^p$ from the known $z_0 \in \mathbb{R}^{p+q}$. We associate equation (41) with the mapping

$$\Phi(\xi) \stackrel{\text{def}}{=} x_0 - h_\xi(y_0 - g(t_0,\xi),t_0,t_0). \tag{42}$$

Obviously, the root of equation (41) is a fixed point of the map Φ, and vice versa. Thus, it is sufficient to prove that there exists a $\delta_2 > 0$ such that for $(t_0,z_0) \in H(\delta_2)$ and $||\xi|| \leqslant \delta_2$ the map Φ has a fixed point. To prove this, let us consider the ball

$$B = \{\xi :\ ||\xi - x_0|| \leqslant ||y_0 - g(t_0,x_0)||\}$$

and show that for sufficiently small $||\xi||$, the map Φ takes B into itself. Applying the Lipschitz conditions for h_ξ and g to show that

$$||\Phi(\xi) - x_0|| = ||h_\xi(y_0 - g(t_0,\xi),t_0,t_0)|| \overset{(19)}{\leqslant} L_\xi||y_0 - g(t_0,\xi)||$$

$$\leqslant L_\xi\big(||y_0 - g(t_0,x_0)|| + ||g(t_0,x_0) - g(t_0,\xi)||\big) \overset{(8.2)}{\leqslant} L_\xi\big(||y_0 - g(t_0,x_0)|| + ||x_0 - \xi||\big)$$

(taking the definition of the ball B into account)

$$\leqslant 2L_\xi||y_0 - g(t_0,x_0)||.$$

Since $L_\xi \to 0$ as $||\xi|| \to 0$, there exists a $\delta_2 > 0$ such that for $||\xi|| \leqslant \delta_2$, the inequality $L_\xi \leqslant 1/2$ holds, and at the same time we can immediately assume that $\delta_2 \leqslant \bar{\delta}(L)$, where $\bar{\delta}(L)$ is taken from (36). Then

$$||\Phi(\xi) - x_0|| \leqslant ||y_0 - g(t_0, x_0)||,$$

which means that the continuous mapping Φ takes the closed ball B into itself. It follows from the Bohl–Brauer theorem that Φ has a fixed point in B. This proves Lemma 9. ■

Lemma 9 completes the proof of Theorem 2. ■

To characterise the neighborhood $H(\delta_2)$ for which the existence of an invariant foliation is proved, we can make the following remark.

Remark 3. The set

$$\widetilde{H}(\delta_2) \stackrel{\text{def}}{=} \{(t, x, y):\ ||y - g(t, x)|| \leqslant \frac{\alpha}{4}||x||^{\frac{m}{k}},\ ||y|| \leqslant ||x|| \leqslant \delta_2, t \in \mathbb{R}\}$$

lies entirely in the $H(\delta_2)$.

Indeed, let $(t_0, x_0, y_0) \in \widetilde{H}(\delta_2)$. Then, if $\xi = x_0$, we have

$$||y_0 - y(t_0, t_0, x_0)|| = ||y_0 - g(t_0, x_0)|| \leqslant \frac{\alpha}{4}||x_0||^{\frac{m}{k}} = \frac{\alpha}{4}||x(t_0, t_0, x_0)||^{\frac{m}{k}}.$$

This means that $(t_0, x_0, y_0) \in H(t_0, x_0) \subset H(\delta_2)$. Thus, $\widetilde{H}(\delta_2) \subset H(\delta_2)$.

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