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THE GAME-THEORETIC APPROACH TO OPTIMAL CHOICE OF THE COST FUNCTION*)

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1. A characteristic feature of a market economy is emission of shares and other securities. The main reason for emission is connected with the necessity of drawing extra means for production development, for technology improvement, and changing the organization structure of enterprises. Moreover, region administrations emit securities. It is important to observe that, buying securities, organizations as well as individuals have different tendency to risk. In the second and third parts of the present article, we consider models of rational behavior of an emitter in converting securities on the market.

2. Suppose that there is an emitter who emits one-type shares; moreover, the cost of one share $p(u)$ depends on the quantity of purchased shares u, $u \in N$, where $N = \{0, 1, 2, 3, \dots\}$ is the set of nonnegative integers. A close statement was considered in [1, 2]. The function $p(u)$, $p(u) \in N$, is monotone decreasing and there are constants p_0 and p_1 such that the following inequalities hold for every $u \in N$:

$$p_0 \leqslant p(u) \leqslant p_1. \tag{1}$$

All such functions constitute the set F.

Suppose that we have n, $i = \overline{1:n}$, purchasers of shares. Each purchaser decides how many shares u_i, $i = \overline{1:n}$, he will purchase for the cost function $p(u_i) \in F$ chosen by the emitter. Each purchaser has his own profit function $S_i(u_i)$. The functions $S_i(u_i)$ are nonnegative and concave,

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$S_i(0) = 0$, and $\lim S_i'(u_i) \leqslant 0$ as $u_i \to \infty$, $i = \overline{1:n}$. It is easily seen that the payoff $H_i(u_i)$ of the ith purchaser in buying u_i shares is equal to

$$H_i(p, u_i) = S_i(u_i) - u_i p(u_i), \quad i = \overline{1:n}. \tag{2}$$

The payoff of the emitter H_0 equals

$$H_0(p, u_1, \dots, u_n) = \sum_{i=1}^{n} u_i p(u_i). \tag{3}$$

Thus, we have the game Γ with $n+1$ players. The payoff functions of the players are defined by formulas (2) and (3). The set of strategies of the ith player (purchaser), $i = \overline{1:n}$, is $X_i = \{u_i \in N : S_i(u_i) - u_i p_0 \geqslant 0\}$. It is obvious that the set X_i is finite. The set of strategies of the emitter (the 0th player) is the set $F_0 = \{p(u) : p(u) \in F, u \in \bigcup_{i=1}^{n} X_i\}$ which is finite as well. Then we have the game $\Gamma = (I; F_0, X_i; H_0, H_i;\ i = \overline{1:n})$, where $I = \{0, 1, \dots, n\}$.

The above-constructed game is a game of "leader (emitter)—wingman" type. In this game we suppose that the purchasers act independently. We now generalize the concept of the Stackelberg equilibrium [3] as follows: We say that $(p^*, u_1^*, \dots, u_n^*)$ is an equilibrium in the game Γ if the following conditions are satisfied:

$$BR = \{(q, x_1, \dots, x_n) : q \in F_0,\ x_i = \arg \sup_{u_i \in X_i} H_i(q, u_i),\ i = \overline{1:n}\},$$

$$H_0(p^*, u_1^*, \dots, u_n^*) = \sup_{(q, x_1, \dots, x_n) \in BR} H_0(q, x_1, \dots, x_n).$$

Theorem 1. *The game Γ possesses an equilibrium.*

The assertion follows from finiteness of the strategy sets.

If we remove the discreteness assumption then we may show that the strategy sets are compact. Assuming that the functions introduced above are continuous, we obtain an analogous result on existence of an equilibrium point.

3. Consider some concrete implementation of our approach to optimal of the cost function. The main difficulty in using the above approach consists in the fact that the emitter does not know the profit functions of the purchasers. The exact values of the profit functions are unavailable for the seller. In this connection, we arrive at the problem of rational choice of the profit functions under deficiency of information.

The loss function of a purchaser has the form

$$\Phi(u) = up(u). \tag{4}$$

If $u_1 < u_2$ then we naturally suppose that the values of loss functions satisfy the inequality

$$\Phi(u_1 + h) - \Phi(u_1) \geqslant \Phi(u_2 + h) - \Phi(u_2)$$

for all $h > 0$.

Henceforth, we suppose that the loss function $\Phi(u)$ is concave; in this case the above inequality holds obviously. Thus, we impose one more constraint on the cost function $p(u)$.

Suppose that the ith purchaser can be described by two parameters: the parameter C_i determines the expected profit of the ith purchaser from one share (the subjective estimate of the expected profit) and the parameter A_i determines the desired total profit of the purchaser from purchasing shares of a given emitter. Then the value $u_i' = A_i/C_i$ determines the quantity of shares the ith purchaser would like to purchase.

It is obvious that the parameter C_i characterizes the optimism of the $i$the purchaser with respect to shares of a given emitter and the parameter A_i characterizes the financial possibilities of the ith purchaser.

Thus, suppose that the profit function of the $i$the purchaser is determined as follows: $S_i(u)$ equals $C_i u$ if $u \leqslant u_i'$ and $S_i(u)$ equals A_i if $u > u_i$.

It is obvious that

$$u_i' = \arg \sup_{u_i \in X_i} H_i(\Phi, u_i),$$

where $H_i(\Phi, u_i) = S_i(u_i) - \Phi(u_i)$ is the value of the expected profit from u_i shares from the viewpoint of the ith purchaser.

Accept the following conjecture characterizing the behavior of the purchasers: the ith purchaser buys u_i' share if and only if

$$H_i(\Phi, u_i') \geqslant qS_i(u_i'),$$

where q is a constant that can be found experimentally, $0 < q < 1$.

Let $Q_{u',A}(x, y)$ be the probability measure presenting the distribution of the random vector (u', A). We may interpret this probability measure as the distribution of purchasers or the distribution characterizing incompleteness of information about the purchasers; in the latter case there may be several purchasers on the market or even a single purchaser.

The expected profit for the emitter from one purchaser is determined by the formula

$$H^0(\Phi, Q_{u',A}) = \int_{\{(x,y):y-\Phi(x)\geqslant qy\}} \Phi(x)dQ_{u',A}(x,y), \tag{5}$$

where $\Phi(x) = xp(x)$ and $p(x)$ is the cost function.

We have thus arrived at the following optimization problem:

$$\begin{aligned} \Phi^* &= \arg\sup H^0(\Phi, Q_{u',A}), \\ p^*(u) &= \Phi^*(u)/u. \end{aligned} \tag{6}$$

We seek a maximum on the set of nondecreasing concave functions Φ such that $p(u) = \Phi(u)/u \in F$.

If the support of the measure $Q_{u',A}$ is finite then there are finitely many admissible functions $\Phi(u)$. In this case the problem (6) can be solved by finite exhaustion of all admissible loss functions.

Conclusion. The idea of a "floating" cost of the unit of a commodity depending on the quantity of the purchased commodity is well known in economics and widely spread in practice, for instance, in converting shares by some Russian banks. However, choice of a cost as a function of the purchased quantity has been of a purely empirical nature; therefore, consideration of this problem from the theoretic viewpoint enables us to outline approaches to optimal choice of the cost function which would considerably increase the economical effect of emission of shares and other securities.

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