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COMMUNICATIONS

ROTA–BAXTER OPERATORS OF NONZERO WEIGHT ON A COMPLETE LINEAR LIE ALGEBRA OF ORDER TWO

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1. PRELIMINARY INFORMATION

Rota–Baxter operators for associative algebras appeared in Baxter's paper [1] as part of the study of integral operators emerging in probability theory and mathematical statistics. Independently, in the early 1980s, Rota–Baxter operators on Lie algebras naturally appeared in [2] on the one hand and in [3] on the other hand, in exploring solutions for the Yang–Baxter equation, one of the most important in the moment equations in mathematical physics. In [4], it was stated that there is a relationship between Rota–Baxter operators of nonzero weight on Lie algebras and non-skew-symmetric solutions to the Yang–Baxter equation, whose symmetric part is *ad*-invariant. Furthermore, by that time it had been found out that Rota–Baxter operators have deep connections with number theory, operad theory, and, in particular, with pre- and post-algebras.

Definition. Let A be an arbitrary algebra over a field F, $R : A \to A$ be a linear mapping, and $\lambda \in F$ be a scalar. The mapping R is called a *Rota–Baxter operator of weight* λ if, for any $x, y \in A$,

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy).$$

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Note that in the case $\lambda \neq 0$ we can consider operators of weight 1, since the operator αR is a Rota–Baxter operator of weight $\alpha\lambda$ for any $\alpha \neq 0$. Thus, if we multiply by a scalar we can obtain any nonzero weight. This comment reduces the study of Rota–Baxter operators to two different cases, one of zero weight and one of nonzero weight.

Below is a well-known assertion, which gives important examples of Rota–Baxter operators of nonzero weight on an arbitrary algebra A.

Assertion 1. Let A_1 and A_2 be subalgebras of A, $A_1 \bigcap A_2 = 0$, and $A = A_1 \bigoplus A_2$. Suppose also that R is a projection operator on A_1 parallel to A_2, i.e., $R(a_1 + a_2) = a_1$ for any $a_1 \in A_1$ and $a_2 \in A_2$. Then R is a Rota–Baxter operator of weight -1 on the algebra A.

Rota–Baxter operators such as in Assertion 1 are said to be *splitting*.

An important problem in this area is describing Rota–Baxter operators on various algebras. In particular, the Rota–Baxter operators on the algebra $sl_2(\mathbb{C})$ were dealt with in [5-7], and those on the matrix algebra $M_2(\mathbb{C})$, in [8, 9]. The classification of Rota–Baxter operators of nonzero weight on $sl_3(\mathbb{C})$ is due to Sokolov [10]. Nonsplitting Rota–Baxter operators of nonzero weight on the matrix algebra $M_3(F)$, where F is an algebraically closed field of characteristic 0, were taken up in [11].

Let A be an arbitrary algebra, and let $R : A \mapsto A$ be a Rota–Baxter operator of arbitrary weight λ and φ be an automorphism or antiautomorphism of the algebra A. Then the operator $\varphi \circ R \circ \varphi^{-1}$ is again a Rota–Baxter operator of the same weight λ on A. This means that Rota–Baxter operators on A can be described up to the action of a group generated by automorphisms and antiautomorphisms of A.

2. MAIN RESULT

In this paper, as a Lie algebra L we take a complete linear Lie algebra $gl_2(F) = (M_2(F), [\cdot,\cdot])$ over an algebraically closed field F with Lie multiplication

$$[x, y] = xy - yx.$$

The objective is to describe Rota–Baxter operators of weight 1 on $gl_2(F)$. Note that if the mapping φ is an antiautomorphism of a Lie algebra, then $-\varphi$ is an automorphism of the same algebra. Thus we will conduct our classification up to the action of the automorphism group $\mathrm{Aut}\,(gl_2(\mathbb{C}))$.

We use the following notation: $\mathrm{E} \in gl_2(\mathbb{C})$ is the identity matrix, e_{ij} are the usual matrix units, and $h = e_{11} - e_{22}$. As a basis for the algebra $gl_2(\mathbb{C})$ we take the set $\mathrm{E}, h, e_{12}, e_{21}$.

Note that if $R(\mathrm{E})_J$ is a Jordan form of the matrix $R(\mathrm{E})$, and T is a transition matrix, then the mapping $\varphi_T : gl_2(\mathbb{C}) \mapsto gl_2(\mathbb{C})$, which acts as

$$\varphi_T(A) = T^{-1}AT,$$

is an automorphism of the matrix algebra $gl_2(\mathbb{C})$. Also

$$\varphi_T \circ R \circ \varphi_{T^{-1}}(\mathrm{E}) = T^{-1}R(T\mathrm{E}T^{-1})T = T^{-1}R(\mathrm{E})T = R(\mathrm{E})_J.$$

Consequently, up to the action of a group generated by automorphisms, we may assume that $R(\mathrm{E})$ is in Jordan form. For $R(\mathrm{E})_J$, we have the following options:

$R(\mathrm{E}) = \lambda E + e_{12}$, $\lambda \in \mathbb{C}$, is a Jordan box of size 2;

$R(\mathrm{E}) = \lambda_1 e_{11} + \lambda_2 e_{22}$, $\lambda_1 \neq \lambda_2 \in \mathbb{C}$, are two boxes corresponding to different eigenvalues;

$R(\mathrm{E}) = \lambda \mathrm{E}$, $\lambda \in E$, are two boxes corresponding to one eigenvalue.

The main results of the paper are two theorems below.

THEOREM 1. Let R be a Rota–Baxter operator of weight 1 on a complete linear Lie algebra $gl_2(\mathbb{C})$. Then, up to the action of the automorphism group, R equals one of the following operators:

$$R(\mathrm{E}) = \lambda\mathrm{E} + e_{12},\ R(h) = R(e_{12}) = R(e_{21}) = 0; \tag{1}$$

$$R(\mathrm{E}) = \lambda\mathrm{E} + e_{12},\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21}, R(h) = -h; \tag{2}$$

$$R(\mathrm{E}) = \lambda\mathrm{E} + h,\ R(h) = 0,\ R(e_{12}) = R(e_{21}) = 0; \tag{3}$$

$$R(\mathrm{E}) = \lambda\mathrm{E} + h,\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21}; \tag{4}$$

$$R(\mathrm{E}) = \lambda\mathrm{E} + h,\ R(h) = \alpha_1 E + \alpha_2 h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0; \tag{5}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = R(e_{21}) = 0,\ R(e_{12}) = -e_{12} + th,\ t \in \{0, 1\}; \tag{6}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = R(e_{21}) = 0,\ R(e_{12}) = -e_{12} + th + \mathrm{E},\ t \in \{0, 1\}; \tag{7}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = \mathrm{E},\ R(e_{12}) = -e_{12} + h + \alpha\mathrm{E};\ R(e_{21}) = 0; \tag{8}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = \mathrm{E},\ R(e_{12}) = -e_{12} + \mathrm{E},\ R(e_{21}) = 0; \tag{9}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th,\ R(e_{21}) = 0,\ R(e_{12}) = -e_{12},\ t \in \mathbb{C}^*; \tag{10}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th + \mathrm{E},\ R(e_{21}) = 0,\ R(e_{12}) = -e_{12},\ t \in \mathbb{C}^*; \tag{11}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = -h + \alpha\mathrm{E},\ R(e_{21}) = \mathrm{E},\ R(e_{12}) = -e_{12}; \tag{12}$$

$$R(\mathrm{E}) = \lambda\mathrm{E}, R(h) = th,\ R(e_{12}) = te_{12},\ R(e_{21}) = te_{21},\ t \in \{0, -1\}. \tag{13}$$

Here $\lambda, \alpha, \alpha_i \in \mathbb{C}$.

THEOREM 2. Operators (1)-(13) lie in different orbits under the action of the automorphism group of the algebra $gl_2(\mathbb{C})$.

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