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Discretization of the Functionals With Prescribed Derivative *

Aleksandr I. Belov *, **Irina V. Alexandrova** *

* *St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg, 199034, Russia (e-mail: st062146@student.spbu.ru,i.v.aleksandrova@spbu.ru).*

Abstract: There are a number of recent results aimed at developing finite necessary and sufficient stability tests for linear time-invariant time delay systems within the framework of Lyapunov-Krasovskii functionals with prescribed derivative approach. These tests are based on the concept of Lyapunov matrix. They usually require verification of positive definiteness of a certain block matrix whose blocks correspond to the Lyapunov matrix evaluated at several discretization points. In this work, we present a new test of this kind which employs a combination of the discretized Lyapunov functionals method of K. Gu and the functionals with prescribed derivative approach. Unlike existing combinations of those techniques, we avoid discretization of the functional's derivative and use the methodology of the necessary and sufficient stability tests development instead. We show a connection between our test and some of the existing results.

Keywords: delay, linear systems, exponential stability, Lyapunov-Krasovskii functionals, Lyapunov matrices, discretized Lyapunov functionals, finite criteria

1. INTRODUCTION

Starting from the seminal work of Krasovskii (1956), the Lyapunov–Krasovskii functionals approach became a powerful tool for the stability analysis of time delay systems in the past half-century. Two basic requirements on functionals within the approach include their positive definiteness and negative definiteness of their derivatives along the solutions of a system under study. As a result, two groups of techniques were emerged.

In the first of them, it is suggested to prescribe the structure of the functionals in advance, the idea that results in sufficient conditions for stability of linear time delay systems in the form of linear matrix inequalities (LMIs; see Fridman, 2014; Seuret & Gouaisbaut, 2015, etc.) In contrast, in the second group of techniques the time derivatives of the functionals are prescribed (see Huang, 1989; Infante & Castelan, 1978; Kharitonov & Zhabko, 2003; Repin, 1966, and also Kharitonov (2013) for significant developments in the theory) thus leading to necessary and sufficient stability conditions of linear time-invariant systems in terms of the functionals.

The discretized Lyapunov functionals method is outstanding representative of the first group (Gu, 1997, 1999, 2001, see also Gu et al. (2003)). The basic idea of the method is to use the Lyapunov–Krasovskii functional of general structure with piecewise linear kernels under the integrals. It appears that such functional admits a tight lower bound in the form of an integral of a quadratic form with a constant matrix. By discretization and estimation of its time derivative, sufficient stability conditions in the form of LMIs are then obtained. It is shown that quite good performance can be achieved in practice using a finer discretization.

On the other hand, an interest to developing the necessary and sufficient finite stability tests within the second group of techniques has grown recently (Alexandrova, 2022; Bajodek et al., 2021; Gomez et al., 2019, 2021). These tests are expressed via the Lyapunov matrix which completely determines the functionals with prescribed derivative and serves as an analogue of a solution of the classical Lyapunov matrix equation for delay free linear differential systems. The core idea behind those tests is to use various types of approximations for an argument of the functionals, namely, a piecewise linear approximation in Alexandrova (2022), a Legendre polynomials based-approximation in Bajodek et al. (2021), a special type of the fundamental matrix-based approximation in Gomez et al. (2019, 2021). It usually appears that replacing the argument by approximation gives the representation of the functional as a quadratic form with a certain constant matrix which is based on the Lyapunov matrix. Meanwhile, the error of the approximation can be bounded. This step is sufficient to be made just for functions from a special set which constitutes an analogue of Razumikhin condition (Alexandrova & Zhabko, 2019; Medvedeva & Zhabko, 2015). Finally, based on the ideas of Gomez et al. (2019) and Medvedeva & Zhabko (2015), dimension of the matrix which corresponds to the necessary and sufficient positive definiteness test is calculated.

In this paper, we develop a new necessary and sufficient stability test of this kind at the juncture between the approach of Lyapunov–Krasovskii functionals with pre-

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scribed derivative and the discretized Lyapunov functionals method of K. Gu. We mention that a combination of these approaches was already made in Mondié & Kharitonov (2004); Ochoa & Mondié (2006, 2007). However, discretization of the derivative was performed in those works along with discretization of the functional itself, resulting in a not very efficient set of LMIs. The main novelty of our work is that we avoid discretization of the functionals derivative, and apply the methodology of developing the necessary and sufficient stability tests described above. In particular, we estimate the difference between the exact functional and the discretized one which tends to zero when discretization is refined. As a result, stability test is reduced to nonnegative definiteness verification of a certain matrix whose dimension is calculated explicitly.

Notation. Let $PC([-h,0],\mathbb{R}^n)$ stand for the space of $\mathbb{R}^n$-valued piecewise continuous functions on $[-h,0]$ equipped with the norm $\|\varphi\|_h = \sup_{\theta\in[-h,0]}\|\varphi(\theta)\|$, and $\mathbb{N}$ be the set of natural numbers. Notation $X > 0$ ($X \geqslant 0$) means that X is a positive (nonnegative) definite matrix; $\lambda_{\min}(W)$ stands for the minimal eigenvalue of a symmetric matrix W; $k = \overline{n_1, n_2}$ means $k = n_1, \dots, n_2$, where $n_1, n_2 \in \mathbb{N}$, and $K = \|A_0\| + \|A_1\|$.

2. PRELIMINARIES

In this work, we consider a linear time delay system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad t \geqslant 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, A_0, A_1 are constant $n\times n$ matrices, and $h \geqslant 0$ is a delay. Denote by $x_t\colon \theta \to x(t+\theta)$, $\theta\in[-h,0]$, the state of system (1).

2.1 Discretized Lyapunov Functionals Method of K. Gu

In the set of works Gu (1997, 1999, 2001) the so-called discretized Lyapunov functionals method was proposed. The idea is to use the Lyapunov–Krasovskii functional of general structure,

$$\begin{aligned} v(\varphi) &= \varphi^T(0)P\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} Q(s)\varphi(s)\mathrm{d}s \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(s_1)R(s_1,s_2)\varphi(s_2)\mathrm{d}s_2\mathrm{d}s_1 + I_4, \end{aligned} \tag{2}$$

$$I_4 = \int_{-h}^{0} \varphi^T(s)W(s)\varphi(s)\mathrm{d}s,$$

where $P = P^T$, $R^T(s_1,s_2) = R(s_2,s_1)$, $W(s) = W^T(s)$, and matrix functions $Q(s)$, $R(s_1,s_2)$, and $W(s)$ are assumed to be piecewise linear. These piecewise linear functions are defined as follows:

$$\begin{aligned} &Q(\theta_p + \alpha\tau) = (1-\alpha)Q_p + \alpha Q_{p-1}, \\ &W(\theta_p+\alpha\tau) = (1-\alpha)W_p + \alpha W_{p-1}, \quad \alpha\in[0,1], \\ &R(\theta_p+\alpha\tau, \theta_q+\beta\tau) \\ &\quad = \begin{cases} (1-\alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha-\beta)R_{p-1,q}, & \alpha \geqslant \beta, \\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta-\alpha)R_{p,q-1}, & \alpha < \beta, \end{cases} \\ &\qquad\qquad \alpha,\beta\in[0,1], \quad p,q = \overline{1,N}, \end{aligned} \tag{3}$$

with $Q_p = Q(\theta_p)$, $W_p = W(\theta_p)$, $R_{pq} = R(\theta_p,\theta_q)$, where $\theta_p = -p\tau$, $p = \overline{0,N}$, are the discretization points of the interval $[-h,0]$, and $\tau = h/N$.

After some nice manipulations based on the integration by parts made in Gu (1997), it is obtained that

$$v(\varphi) = \int_0^1 \left(\varphi^T(0)\Psi^T(\alpha)\right)\begin{pmatrix} P & \mathcal{Q}_N \\ \mathcal{Q}_N^T & \mathcal{R}_N\end{pmatrix}\begin{pmatrix}\varphi(0)\\ \Psi(\alpha)\end{pmatrix}\mathrm{d}\alpha + I_4,$$

$$I_4 \geqslant \int_0^1 \Psi^T(\alpha)\mathcal{W}_N\Psi(\alpha)\mathrm{d}\alpha,$$

where

$$\mathcal{Q}_N = \Big(Q_0, Q_1, \dots, Q_N\Big), \quad \mathcal{R}_N = \Big\{R_{pq}\Big\}_{p,q=0}^N,$$

$$\mathcal{W}_N = \frac{1}{\tau}\begin{pmatrix} W_0 & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{O} & W_1 & \dots & \mathbb{O} \\ \vdots & & \ddots & \\ \mathbb{O} & \mathbb{O} & \dots & W_N \end{pmatrix}.$$

The vector $\Psi(\alpha)$ is of the form

$$\Psi(\alpha) = \Big(\psi_0^T(\alpha), \psi_1^T(\alpha), \dots, \psi_N^T(\alpha)\Big)^T,$$

where

$$\begin{aligned} \psi_0(\alpha) &= \tau\int_\alpha^1 \varphi(\theta_1 + s\tau)\mathrm{d}s, \\ \psi_j(\alpha) &= \tau\left(\int_\alpha^1 \varphi(\theta_{j+1}+s\tau)\mathrm{d}s \right. \\ &\qquad \left. + \int_0^\alpha \varphi(\theta_j + s\tau)\mathrm{d}s\right), \quad j = \overline{1, N-1}, \\ \psi_N(\alpha) &= \tau\int_0^\alpha \varphi(\theta_N + s\tau)\mathrm{d}s. \end{aligned}$$

Next, the functional is differentiated along the solutions of system (1),

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t} \leqslant -\xi^T(x_t)\Theta_N\xi(x_t), \quad t\geqslant 0,$$

where $\xi(\varphi)$ is a special vector, and Θ_N is a constant matrix. Therefore, the following result is suggested.

Theorem 1. Gu (1997) If matrices

$$\begin{pmatrix} P & \mathcal{Q}_N \\ \mathcal{Q}_N^T & \mathcal{R}_N + \mathcal{W}_N \end{pmatrix} > 0, \quad \Theta_N > 0,$$

then system (1) is exponentially stable.

2.2 Functionals with Prescribed Derivative and Lyapunov Matrix

A particular structure of functional (2) which corresponds to the necessary and sufficient stability condition of system (1) is known. Below, we summarize the constructions of so-called functionals with prescribed derivative developed mainly in Huang (1989); Infante & Castelan (1978); Kharitonov & Zhabko (2003).

The matrix function $U(\theta)$ is called the Lyapunov matrix of system (1) associated with $W = W^T$, if it is a solution of the dynamic equation

$$\frac{\mathrm{d}U(\theta)}{\mathrm{d}\theta} = U(\theta)A_0 + U(\theta-h)A_1, \quad \theta > 0,$$

which satisfies in addition the symmetry and algebraic properties

$$U(-\theta) = U^T(\theta), \quad \theta \geqslant 0,$$
$$U'(+0) - U'(-0) = -W.$$

This matrix is the main element which determines functionals with prescribed derivative. In this paper, we make use of the simplest functional with prescribed derivative which is of the form

$$v_0(\varphi) = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U^T(h+s)A_1\varphi(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(s_1)A_1^T U(s_1-s_2)A_1\varphi(s_2)\mathrm{d}s_2\mathrm{d}s_1 \quad (4)$$

and was introduced in Huang (1989) for the first time. The main feature of this functional is that along the solutions of system (1) it satisfies

$$\frac{\mathrm{d}v_0(x_t)}{\mathrm{d}t} = -x^T(t)Wx(t), \quad t \geqslant 0.$$

Here, W is a positive definite matrix. However, as it was shown in Huang (1989), functional (4) admits only a local cubic lower bound, and thus does not satisfy conditions of the classical stability theorem, see, for instance, Theorem 1.9 in Kharitonov (2013). Some useful modifications of functional (4) were introduced later in Kharitonov & Zhabko (2003) and Egorov & Mondié (2014). In fact, the modifications are made by adding the summand I_4 with linear and constant form of the matrix $W(s)$, respectively. They provide the necessary and sufficient stability conditions expressed in terms of the functionals with prescribed derivative. Alternatively, by introducing the set of functions

$$S = \Big\{\varphi \in PC\big([-h,0],\mathbb{R}^n\big) \;\Big|\; \|\varphi\|_h = \|\varphi(0)\| = 1\Big\},$$

the necessary and sufficient stability conditions based on original functional (4) were presented in Medvedeva & Zhabko (2015) and Alexandrova & Zhabko (2019):

Theorem 2. System (1) is exponentially stable, if and only if there exist functional (4) and $\alpha_1 > 0$ such that

$$v_0(\varphi) \geqslant \alpha_1, \quad \varphi \in S.$$

Moreover, if system (1) is unstable and functional (4) exists, then there is a function $\widetilde{\varphi} \in S$ such that

$$v_0(\widetilde{\varphi}) \leqslant -a_0,$$

where $a_0 = \dfrac{\lambda_{\min}(W)}{4K}$. Here, $K = \|A_0\| + \|A_1\|$.

It is worth noting that necessary and sufficient stability conditions expressed exclusively via the Lyapunov matrix are known, see Gomez et al. (2019). In this paper, we make use of their necessity part which is as follows.

Lemma 3. (Egorov & Mondié, 2014) If system (1) is exponentially stable, then for any $N \in \mathbb{N}$ matrix

$$\mathcal{K}_N = \{U((j-i)\tau)\}_{i,j=0}^N > 0.$$

We also mention that a necessary and sufficient condition for the existence and uniqueness of the Lyapunov matrix is the so-called Lyapunov condition, i.e. the absence of the eigenvalues of system (1) located symmetrically with respect to the origin of the complex plane. This condition may be verified numerically during the computation of the Lyapunov matrix.

2.3 Schur Complement

Introduce some auxiliary statements on the matrix properties.

Let

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}.$$

Lemma 4. (Schur complement) The following statements are true:

(a). Let $A > 0$. Then, $X \geqslant 0$, if and only if

$$C - B^TA^{-1}B \geqslant 0.$$

(b). Matrix $X > 0$, if and only if $C > 0$ and

$$A - BC^{-1}B^T > 0.$$

3. DISCRETIZATION OF THE FUNCTIONALS WITH PRESCRIBED DERIVATIVE

3.1 Discretization Process

Let us apply the discretization process of Gu (1997, 1999, 2001) described in Section 2.1 to functional (4). The first step is to simply replace the kernels

$$Q(s) = U^T(h+s)A_1, \quad R(s_1,s_2) = A_1^TU(s_1-s_2)A_1 \quad (5)$$

by their piecewise linear approximations as it is done in Mondié & Kharitonov (2004). Then, we obtain the discretized version of functional (4) which can be treated as the approximation for the functional:

$$v_0^{(N)}(\varphi) = \int_0^1 \big(\varphi^T(0)\Psi^T(\alpha)\big)\begin{pmatrix} U(0) & \mathcal{Q}_N \\ \mathcal{Q}_N^T & \mathcal{R}_N \end{pmatrix}\begin{pmatrix} \varphi(0) \\ \Psi(\alpha) \end{pmatrix} d\alpha. \quad (6)$$

Here,

$$\mathcal{Q}_N = \big(U^T(N\tau)A_1;\ \ldots\ U^T(\tau)A_1;\ U(0)A_1\big),$$
$$\mathcal{R}_N = \Big\{A_1^TU((j-i)\tau)A_1\Big\}_{i,j=0}^N.$$

3.2 A necessary condition

It appears that nonnegative definiteness of a matrix which defines discretized functional (6) forms a necessary condition for the exponential stability of system (1).

Lemma 5. If system (1) is exponentially stable, then for any $N \in \mathbb{N}$ matrix

$$\mathcal{A}_N = \begin{pmatrix} U(0) & \mathcal{Q}_N \\ \mathcal{Q}_N^T & \mathcal{R}_N \end{pmatrix} \geqslant 0. \quad (7)$$

Proof. It follows from Lemma 3 that $U(0) > 0$ and $\mathcal{K}_N > 0$ for any $N \in \mathbb{N}$. We present the latter matrix in the form

$$\mathcal{K}_N = \begin{pmatrix} \mathcal{K}_{N-1} & \mathcal{B}_N \\ \mathcal{B}_N^T & U(0) \end{pmatrix},$$

where

$$\mathcal{K}_{N-1} = \{U((j-i)\tau)\}_{i,j=0}^{N-1}, \quad \mathcal{B}_N = \begin{pmatrix} U(N\tau) \\ \vdots \\ U(\tau) \end{pmatrix},$$

and apply proposition (b) of Lemma 4:

$$\mathcal{S} = \mathcal{K}_{N-1} - \mathcal{B}_N U^{-1}(0)\mathcal{B}_N^T = \{\mathcal{S}_{ij}\}_{i,j=0}^{N-1} > 0,$$

where

$$\mathcal{S}_{ij} = U((j-i)\tau) - U((N-i)\tau)U^{-1}(0)U^T((N-j)\tau).$$

Now, construct Schur complement of the block $U(0)$ of matrix (7):

$$\mathcal{R}_N - \mathcal{Q}_N^T U^{-1}(0)\mathcal{Q}_N = \begin{pmatrix} \widetilde{\mathcal{S}} & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{O}_n \end{pmatrix},$$

where

$$\widetilde{\mathcal{S}} = \left\{A_1^T \mathcal{S}_{ij} A_1\right\}_{i,j=0}^{N-1}.$$

Notice that

$$\widetilde{\mathcal{S}} = \begin{pmatrix} A_1^T & \mathbb{O}_n & \dots & \mathbb{O}_n \\ \mathbb{O}_n & A_1^T & \dots & \mathbb{O}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O}_n & \mathbb{O}_n & \dots & A_1^T \end{pmatrix} \mathcal{S} \begin{pmatrix} A_1 & \mathbb{O}_n & \dots & \mathbb{O}_n \\ \mathbb{O}_n & A_1 & \dots & \mathbb{O}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O}_n & \mathbb{O}_n & \dots & A_1 \end{pmatrix} \geqslant 0.$$

Hence, we conclude by proposition (a) of Lemma 4. □

It is worthy of mention that Lemma 5 shows a connection between the matrix $\mathcal{A}_N$ and the necessary stability conditions of Egorov & Mondié (2014).

3.3 A Bound for the Approximation Error

The aim of this section is to demonstrate that functional (4) can be approximated by its discretized version (6) with any predetermined accuracy, at least on the set of functions S. A key result in this direction where we present a constructive bound for the approximation error is given below.

Theorem 6. The following inequality holds for any $N \in \mathbb{N}$ and any $\varphi \in PC([-h,0],\mathbb{R}^n)$:

$$\left| v_0(\varphi) - v_0^{(N)}(\varphi) \right| \leqslant \frac{c}{N} \|\varphi\|_h^2. \quad (8)$$

Here,

$$c = 2ah^2\left(1 + \frac{ah}{3}\right)K\|U\|, \quad a = \|A_1\|,$$
$$\|U\| = \max_{\theta\in[0,h]} \|U(\theta)\|.$$

Proof. In this proof, we use notation (5) for the functional kernels. Denote piecewise linear approximations of $Q(s)$ and $R(s_1,s_2)$ defined in formulae (3) by $Q^N(s)$ and $R^N(s_1,s_2)$, respectively. We have

$$\left| v_0(\varphi) - v_0^{(N)}(\varphi) \right| \leqslant 2\|\varphi\|_h^2 \int_{-h}^0 \left\| Q(s) - Q^N(s) \right\| \mathrm{d}s$$
$$+ \|\varphi\|_h^2 \int_{-h}^0 \int_{-h}^0 \left\| R(s_1,s_2) - R^N(s_1,s_2) \right\| \mathrm{d}s_2 \mathrm{d}s_1. \quad (9)$$

Notice that for any $\xi \in [-h,0)$ the following holds:

$$\|U'(\xi)\| = \left\| -U'^T(-\xi) \right\| = \|U'(-\xi)\|$$
$$= \|U(-\xi)A_0 + U(-\xi-h)A_1\|,$$

hence $\|U'(\xi)\| \leqslant K\|U\|$. From the above it can be easily seen that the same inequality holds for $\xi \in (0,h]$. This implies that for any $l = \overline{-(N-1), N-1}$ and any $\alpha \in (0,1]$ we have

$$\|U((l+\alpha)\tau) - U(l\tau)\| \leqslant K\|U\|\alpha\tau. \quad (10)$$

Let us estimate two integrals in (9). For the first one we have

$$J_1 = \int_{-h}^0 \left\| Q(s) - Q^N(s) \right\| \mathrm{d}s$$
$$= \sum_{p=1}^N \int_{\theta_p}^{\theta_{p-1}} \left\| Q(s) - Q^N(s) \right\| \mathrm{d}s$$
$$= \tau \sum_{p=1}^N \int_0^1 \left\| Q(\theta_p + \alpha\tau) - Q^N(\theta_p + \alpha\tau) \right\| \mathrm{d}\alpha.$$

Substituting the definition of $Q^N(s)$, we get

$$J_1 = \sum_{p=1}^N \int_0^1 \left\| Q(\theta_p + \alpha\tau) - \left(Q_p + \alpha(Q_{p-1} - Q_p)\right) \right\| \tau d\alpha$$
$$\leqslant \tau \sum_{p=1}^N \int_0^1 \Big(\|Q(\theta_p + \alpha\tau) - Q_p\| + \alpha\|Q_{p-1} - Q_p\| \Big) d\alpha.$$

By (10), we obtain

$$\|Q(\theta_p + \alpha\tau) - Q_p\| = \big\| U^T((N-p+\alpha)\tau)A_1$$
$$- U^T((N-p)\tau)A_1 \big\| \leqslant aK\|U\|\alpha\tau,$$

and, in particular,

$$\|Q_{p-1} - Q_p\| = \|Q(\theta_p + \tau) - Q_p\| \leqslant aK\|U\|\tau.$$

Hence, we get

$$J_1 \leqslant \tau^2 aK\|U\| \sum_{p=1}^N \int_0^1 2\alpha d\alpha = \frac{ah^2K\|U\|}{N}.$$

Now, we deal similarly with the second term in (9):

$$J_2 = \int_{-h}^0 \int_{-h}^0 \left\| R(s_1,s_2) - R^N(s_1,s_2) \right\| \mathrm{d}s_2 \mathrm{d}s_1$$
$$= \tau^2 \sum_{p=1}^N \sum_{q=1}^N \int_0^1 \int_0^1 \| R(\theta_p + \alpha\tau, \theta_q + \beta\tau)$$
$$- R^N(\theta_p + \alpha\tau, \theta_q + \beta\tau) \| \mathrm{d}\beta \mathrm{d}\alpha.$$

Dividing the inner integral into two parts and substituting the definition of R^N in each part, we arrive at

$$J_2 = \tau^2 \sum_{p=1}^N \sum_{q=1}^N \int_0^1 \int_0^\alpha \big\| R(\theta_p + \alpha\tau, \theta_q + \beta\tau) - R_{pq}$$
$$+ \alpha(R_{pq} - R_{p-1,q}) + \beta(R_{p-1,q} - R_{p-1,q-1}) \big\| \mathrm{d}\beta \mathrm{d}\alpha$$
$$+ \tau^2 \sum_{p=1}^N \sum_{q=1}^N \int_0^1 \int_\alpha^1 \big\| R(\theta_p + \alpha\tau, \theta_q + \beta\tau) - R_{pq}$$
$$+ \alpha(R_{p,q-1} - R_{p-1,q-1}) + \beta(R_{pq} - R_{p,q-1}) \big\| \mathrm{d}\beta \mathrm{d}\alpha.$$

Next, using $R_{p-1,q-1} = R_{pq}$, we write

$$J_2 \leqslant \tau^2 \sum_{p=1}^N \sum_{q=1}^N \int_0^1 \int_0^\alpha \big(\| R(\theta_p + \alpha\tau, \theta_q + \beta\tau) - R_{pq} \|$$
$$+ (\alpha - \beta) \| R_{pq} - R_{p-1,q} \| \big) \mathrm{d}\beta \mathrm{d}\alpha$$
$$+ \tau^2 \sum_{p=1}^N \sum_{q=1}^N \int_0^1 \int_\alpha^1 \big(\| R(\theta_p + \alpha\tau, \theta_q + \beta\tau) - R_{pq} \|$$
$$+ (\beta - \alpha) \| R_{pq} - R_{p,q-1} \| \big) \mathrm{d}\beta \mathrm{d}\alpha.$$

Substitute the expressions for R and R_{pq} and again use (10):

$$\begin{aligned}&\left\|R(\theta_p+\alpha\tau,\theta_q+\beta\tau)-R_{pq}\right\|\\&\quad=\left\|A_1^T\big(U((-p+q+\alpha-\beta)\tau)-U((-p+q)\tau)\big)A_1\right\|\\&\qquad\leqslant a^2K\,\|U\|\,|\alpha-\beta|\,\tau.\end{aligned}$$

In particular,

$$\left\|R_{p-1,q}-R_{pq}\right\|=\left\|R(\theta_p+\tau,\theta_q)-R_{pq}\right\|\leqslant a^2K\,\|U\|\,\tau,$$
$$\left\|R_{p,q-1}-R_{pq}\right\|=\left\|R(\theta_p,\theta_q+\tau)-R_{pq}\right\|\leqslant a^2K\,\|U\|\,\tau.$$

Finally, we arrive at the bound

$$J_2\leqslant\tau^3a^2K\,\|U\|\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\int_0^1\int_0^\alpha 2(\alpha-\beta)\mathrm{d}\beta\mathrm{d}\alpha\right.$$
$$\left.+\int_0^1\int_\alpha^1 2(\beta-\alpha)\mathrm{d}\beta\mathrm{d}\alpha\right)=\frac{2a^2h^3K\,\|U\|}{3N}.$$

Combining the estimates for J_1 and J_2, we get the result. □

Corollary 7. For any $\varepsilon>0$ and any $N>N(\varepsilon)=\lceil c/\varepsilon\rceil$ we have

$$|v_0(\varphi)-v_0^{(N)}(\varphi)|<\varepsilon,$$

if $\varphi\in S$.

4. MAIN RESULT

Here, we present a necessary and sufficient stability condition for system (1) which is based on the discretization process described in the previous section. To develop this condition, a similar methodology as in Gomez et al. (2019), Bajodek et al. (2021), Alexandrova (2022) is used.

Let us define the value

$$N^\star=1+\left\lceil\frac{c}{a_0}\right\rceil,$$

where a_0 is introduced in Theorem 2.

Theorem 8. System (1) is exponentially stable, if and only if the Lyapunov condition holds and the matrix

$$\mathcal{A}_{N^\star}\geqslant 0.$$

Proof. The necessity part follows by Lemma 5. Let us prove sufficiency. Since $\mathcal{A}_{N^\star}\geqslant 0$, we have

$$v_0^{(N^\star)}(\varphi)=\int_0^1\left(\varphi^T(0)\Psi^T(\alpha)\right)\mathcal{A}_{N^\star}\begin{pmatrix}\varphi(0)\\\Psi(\alpha)\end{pmatrix}\mathrm{d}\alpha\geqslant 0$$

for any function $\varphi\in PC\big([-h,0],\mathbb{R}^n\big)$. Suppose, by contradiction, that system (1) is unstable. Then, according to Theorem 2, there exists a function $\widetilde{\varphi}\in S$ such that

$$v_0(\widetilde{\varphi})\leqslant -a_0.$$

Formula (8) applied for this function implies

$$v_0^{(N^\star)}(\widetilde{\varphi})\leqslant v_0(\widetilde{\varphi})+\frac{c}{N^\star}\leqslant -a_0+\frac{c}{N^\star}.$$

By the choice of $N^\star$, we get $c/N^\star<a_0$, and hence

$$v_0^{(N^\star)}(\widetilde{\varphi})<0,$$

a contradiction.

□

Remark 9. As it is usually considered, a better option is to develop a positive definiteness instead of nonnegative definiteness verification test, since the presence of eigenvalues located closely to zero may lead to not accurate results. In that regard, it should be noted that there exists a constant $\chi>0$ such that condition

$$\lambda_{\min}\left(\mathcal{A}_N\right)\geqslant-\chi$$

already implies the exponential stability of system (1). This constant may be calculated explicitly based on the value a_0 from Theorem 2. In addition, we mention that techniques based on Cholesky decomposition implemented in MATLAB allow us to perform nonnegative definiteness verification in a more accurate manner than the eigenvalue-based techniques.

5. EXAMPLE

Consider a system

$$\dot{x}(t)=\begin{pmatrix}0&1\\0&0\end{pmatrix}x(t)+\begin{pmatrix}0&0\\-b&-a\end{pmatrix}x(t-1),\quad a,b\in\mathbb{R}.$$

Create a uniform mesh on $[0,2]\times[0,0.6]$ and apply Theorem 8 to each system with parameters (a,b) on this mesh. The condition $\mathcal{A}_{N^*}\geqslant 0$ is verified with “cholcov” function in MATLAB. If the system satisfies the Lyapunov condition and $\mathcal{A}_{N^*}\geqslant 0$, the point (a,b) is marked with a blue dot. The simulation results are given on Fig. 1. Here, red lines correspond to D-partition lines. They split the plane (a,b) into domains such that all systems from one domain are either exponentially stable or unstable. On Fig. 1, we can detect the domain of exponentially stable systems.

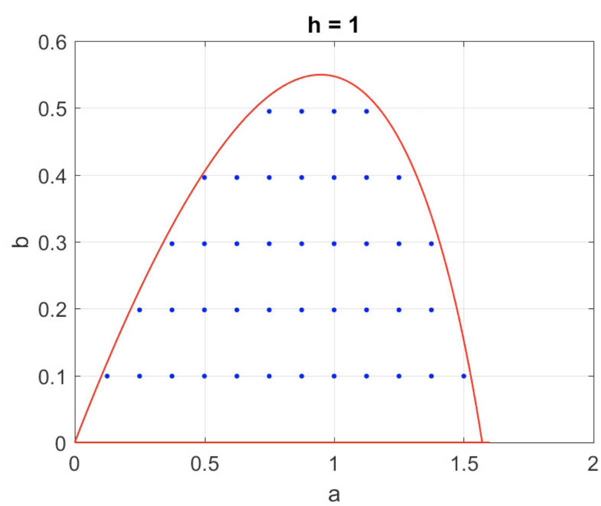

Fig. 1. The exponential stability domain on (a,b)

The results on the dimensions of matrix $\mathcal{A}_{N^*}$ are gathered in Table 1. The average program runtime was 65.7409 sec. for a grid point.

Table 1. Dimension of matrix $\mathcal{A}_{N^*}$

	min	avg	max
size($\mathcal{A}_{N^*}$)	74	1725.4	9972

6. CONCLUSION

A constructive criterion for determining the exponential stability of a linear time-invariant system with one delay is given. It consists in verifying that a certain matrix is positive semidefinite. The idea of the criterion is to replace the kernels in the Lyapunov – Krasovskii functional by their piecewise linear approximations. An example demonstrating the computational performance is given.

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