The Local Nonpolynomial Splines and Solution of Integro-Differential Equations

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Abstract: The application of the local polynomial splines to the solution of integro-differential equations was regarded in the author's previous papers. In a recent paper, we introduced the application of the local nonpolynomial splines to the solution of integro-differential equations. These splines allow us to approximate functions with a presribed order of approximation. In this paper, we apply the splines to the solution of the integro-differential equations with a smooth kernel. Applying the trigonometric or exponential spline approximations of the fifth order of approximation, we obtain an approximate solution of the integro-differential equation at the set of nodes. The advantages of using such splines include the ability to determine not only the values of the desired function at the grid nodes, but also the first derivative at the grid nodes. The obtained values can be connected by lines using the splines. Thus, after interpolation, we can obtain the value of the solution at any point of the considered interval. Several numerical examples are given.

Key-Words: Local nonpolynomial splines, local trigonometric splines, local exponential splines, integro-differential equation, the fifth order of approximation

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1 Introduction

Many mathematical models are described by linear or nonlinear integral equations. Integral equations appear in nonlinear physical phenomenons such as electomagnetic fluid dynamics, reformulation of boundary value problem. Integro-differential equations are encountered in modeling various processes.

Integro-differential equations (IDEs) have been used extensively in biological models, economics, oscillation theory, ocean circulations, control theory of industrial mathematics and other fields [1], [2]. Paper [3] is devoted to the study of an integro-differential system of equations modeling the genetic adaptation of a pathogen by taking into account both the mutation and selection processes. Using the variance of the dispersion in the phenotype trait space as a small parameter the authors provide a complete picture of the dynamical behaviour of the solutions of the problem.

The (2+1) dimensional Konopelchenko–Dubrovsky equation (2D-KDE) is an integro differential equation which describes a two-layer fluid in shallow water near ocean shores and stratified atmosphere (see paper [4]).

The charged particle motion for certain configurations of oscillating magnetic fields can be simulated by a Volterra integro-differential equation of the second order with time-periodic coefficients (see paper [5]).

In study [6], a numerical technique with hybrid approximation is developed for solving high-order linear integro-differential equations including variable delay under the initial conditions. These types of problems are of applications in mathematical physics, mechanics, natural sciences, electronics and computer science.

As noted in paper [7], the wireless sensor network and industrial internet of things have been a growing area of research which is being exploited in various fields such as smart homes, smart industries, smart transportation, and so on. There is a need for a mechanism which can easily tackle the problems of nonlinear delay integro-differential equations for large-scale applications of the Internet of Things. In paper [7], the Haar wavelet collocation technique is developed for the solution of nonlinear delay integro-differential equations for the wireless sensor network and the industrial Internet of Things. The method is applied to the nonlinear delay Volterra, delay Fredholm and delay Volterra–Fredholm integro-differential equations which are based on the use of Haar wavelets.

Paper [8] noted that it is well known that the study of many processes of the natural sciences can be reduced to solving the Volterra integrodifferential equations. Recent studies on certain problems such as the HIV virus, bird flu virus, and diseases associated with mutations of viruses have become relevant. A solution to such problems is associated with finding solutions of VIDEs. There are several classes of methods for solving IDEs. In contrast to the known methods, paper [8] developed the finite difference hybrid method by a combination of power series and the shifted Legendre polynomial through a block method.

At present, many authors are trying to construct more accurate methods for solving integro-differential equations. In paper [9] efficient numerical methods are given to solve the linear Volterra integral equations and Volterra Integro differential equations of the first and second types with exponential, singular, regular and convolution kernels.

In paper [10] the authors introduce a numerical method for solving the nonlinear Volterra integro-differential equations. In the first step, the authors apply the implicit trapezium rule to discretize the integral in a given equation. Further, the Daftardar-Gejji and Jafari technique is used to find the unknown term on the right side.

In study [11], the second order linear Volterra partial integro-differential equation are solved with the collocation method based on the Lerch polynomials.

In paper [12], a 6th order Runge-Kutta with the seven stages method for finding the numerical solution of the Volterra integro-differential equation is considered. Here the integral term in the Volterra integro-differential equation approximated using the Lagrange interpolation numerical method is discussed.

Finite elements, splines and wavelets are often used to construct computational schemes for solving integro-differential equations.

In paper [13], the authors present a new mixed finite element method for a class of parabolic equations with p-Laplacian and nonlinear memory.

In paper [14], a new collocation method based on the Pell–Lucas polynomials is presented to solve the parabolic-type partial Volterra integro-differential equations.

In paper [15], a new three-point linear rational finite difference (3LRFD) formula is investigated, which is combined with the compound trapezoidal scheme to discretize the differential term and integral term of second-order linear Fredholm integro-differential equation (SOLFIDE) respectively, and then the corresponding 3LRFD-quadrature approximation equation can be derived and then generate the large and dense linear system.

In paper [16], the authors consider the Jacobi collocation method for the numerical solution of the neutral nonlinear weakly singular Fredholm integro-differential equations.

Paper [17], focuses on an efficient spline-based numerical technique for numerically addressing a second-order Volterra partial integrodifferential equation. In paper [George] the time derivative is discretized using a finite difference scheme, while the space derivative is approximated using the extended cubic B-spline basis.

Paper [18], aims to present a new method for the approximate solution of two-dimensional nonlinear Volterra–Fredholm partial integro-differential equations with boundary conditions using two-dimensional Chebyshev wavelets.

In paper [19], the authors approximate the solution of Fredholm integro-differential equations of the second kind by using exponential spline function. The proposed method reduces to the system of algebraic equations.

Although two-dimensional (2D) parabolic integro-differential equations (PIDEs) arise in many physical contexts, there is no general available software that is able to solve them numerically. To remedy this situation, in paper [20], the authors provide a compact implementation for solving 2D PIDEs using the finite element method (FEM) on unstructured grids. Piecewise linear finite element spaces on triangles are used for the space discretization, whereas the time discretization is based on the backward-Euler and the Crank–Nicolson methods. The quadrature rules for discretizing the Volterra integral term are chosen so as to be consistent with the time-stepping schemes; a more efficient version of the implementation that uses a vectorization technique in the assembly process is also presented.

In work [21], the Legendre wavelet collocation method is implemented for the numerical solution of nonlinear integral and integro-differential equations. The authors approximate the solution with the Legendre wavelet.

In this paper we consider the solution of the linear Volterra–Fredholm integro-differential equations of the second kind with a continuous kernel and a continuous right-hand side. When solving such equations, the application of polynomial spline approximations is one of the possible ways to calculate a solution in the grid points (see the author's paper [22]). After we have computed the solution at the grid nodes, we can compute the solution at additional points between these nodes using the interpolation with these spline approximations. Moreover, the resulting line can not only be continuous, but it can also be quite smooth. Methods for constructing such approximations were previously considered in the author's papers.

The method for calculating the error of approximation using the non-polynomial splines is given in the author's paper [23]. When solving linear Volterra–Fredholm integro-differential equations of the second kind, the use of non-polynomial splines can give a more accurate result. However, there may be problems with the calculation of the integral. In this case, it is necessary to apply the corresponding quadrature formulas. These quadrature formulas can also be built using the local non-polynomial splines. The second section presents formulas for the trigonometric, exponential and polynomial splines of the fifth order of approximation. The third section presents the results of numerical experiments.

2 Construction of Nonpolynomial Approximations

2.1 Nonpolynomial Approximations of the Fifth Order

The basic splines of the fifth-order approximation are found separately on each grid interval. Recall that when constructing an approximation on a finite interval $[a, b]$, we have to use different types of fifth-order approximations. We have to distinguish between the approximations near the left end of the interval $[a, b]$, the right end of the interval $[a, b]$, and near the middle of the interval $[a, b]$. We note that the approximation with the middle splines gives a smaller approximation error compared to the approximations with the left splines or with the right splines. To do this, we have to solve a system of approximation relations. Suppose that the functions φ_i, $i = 0, 1, 2, 3, 4$, form a Chebyshev system, and the determinant of the system is nonzero. Let the values of the function $u(x)$ be known at the nodes of the grid $\{x_k\}$:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Approximation with the local splines of the fifth order of approximation is built separately on each grid interval $[x_j, x_{j+1}]$. Denote $u_k = u(x_k)$. In the case of the middle basis splines, the system of equations looks as follows:

$$\sum_{k=j-2}^{j+2} \varphi_i(x_k)\, w_k^s(x) = \varphi_i(x),\ x \in [x_j, x_{j+1}], \quad (1)$$
$$i = 0, 1, 2, 3, 4.$$

In the case of the left basis splines, the system of equations looks as follows:

$$\sum_{k=j}^{j+4} \varphi_i(x_k)\, w_j^L(x) = \varphi_i(x),\ x \in [x_j, x_{j+1}], \quad (2)$$

$$i = 0, 1, 2, 3, 4.$$

In the case of the right basis splines, the system of equations looks as follows:

$$\sum_{k=j-3}^{j+1} \varphi_i(x_k)\, w_j^R(x) = \varphi_i(x),\ x \in [x_j, x_{j+1}], \quad (3)$$

$$i = 0, 1, 2, 3, 4.$$

Solving this system of equations, we obtain formulas for the basis splines.

2.2 Trigonometric Splines of the Fifth Order of Approximation

In order to simplify the expressions, we will do the following on the equidistant set of nodes with step h on $[a, b]$. Let us introduce a variable $t \in [0,1]$. Now we get for $x \in [x_j, x_{j+1}]$: $x = x_j + t\,h$. To construct trigonometric splines of the fifth order of approximation, we take

$$\varphi_0 = 1, \varphi_1 = \sin(x), \varphi_2 = \cos(x),$$
$$\varphi_3 = \sin(2x), \varphi_4 = \cos(2x). \quad (4)$$

Let supp $w_j = [x_{j-2}, x_{j+3}]$. Now the middle trigonometric basis splines (according relation (1)) on the interval $[x_j, x_{j+1}]$: can be represented as followed:

$$\begin{gathered} w_j^s(x_j + t\,h) = 1/128\ (-1 - 2\ \cos(h) \\ +4\ \cos(3\ h) - 2\ \cos(9\ h)) - \cos(6\ h) \\ +\cos(8\ h) + \cos(10\ h) - \cos(t\ h - 5\ h) \\ -\cos(t\ h + 5\ h) + \cos(t\ h - 3\ h) \\ +2\ \cos(t\ h + 7\ h) - \cos(t\ h - h) \\ -\cos(t\ h + h) - 2\ \cos(t\ h - 4\ h) \\ -2\ \cos(t\ h + 4\ h) + 2\ \cos(-3\ h + 2\ t\ h) \\ +2\ \cos(3\ h + 2\ t\ h) - 3\ \cos(-h + 2\ t\ h) \\ +2\ \cos(-2\ h + 2\ t\ h) - 2\ \cos(6\ h + 2\ t\ h) \\ +2\ \cos(2\ h + 2\ t\ h) + \cos(2\ h + t\ h) \\ -\cos(t\,h - 6\,h) - \cos(t\,h + 6\,h) \\ -3\ \cos(h + 2\,t\,h) - \cos(t\,h - 9\,h) \end{gathered}$$

$$+\cos(t\ h+3\ h)-2\cos(-6\,h+2\,t\,h)$$
$$+\cos(-2\,h+t\,h)+\cos(t\ h-8\ h)$$
$$+\cos(t\ h+8\ h)+\cos(2\,t\,h-7\ h)$$
$$+2\ \cos(t\,h-7\,h)+2\ \cos(t\,h)$$
$$+\cos(7\,h+2\,t\,h)$$
$$-\cos\,(t\,h+9\,h))/(\sin^4(h)\cos\,(h)(4\cos^5\,(h)$$
$$-8\cos^4(h)+\cos^3(h)+5\cos^2(h)-\cos(h)$$
$$-1)),$$

$$w^s_{j+1}\big(x_j+t\,h\big)=-1/128\ \ (-1+\cos(h)$$
$$-\cos(2\ h))$$
$$+2\ \cos(4\ h)-\cos(3\ h)-2\ \cos(7\ h)$$
$$+\cos(5\ h)$$
$$+\cos(9\ h)+\cos(6\ h)-\cos(8\ h)$$
$$+2\ \cos(t\ h+5\ h)$$
$$-2\ \cos(t\ h-3\ h)-3\ \cos(t\ h-4\ h)$$
$$-\cos(t\ h+4\ h)+3\ \cos(-3\ h+2\ t\ h)$$
$$+\cos(-h+2\ t\ h)-2\ \cos(6\ h+2\ t\ h)$$
$$+\cos(2\ h+2\ t\ h)-2\ \cos(2\ h+t\ h)$$
$$+2\ \cos(t\ h-6\ h)-2\ \cos(t\ h+6\ h)$$
$$-\cos(-4\ h+2\ t\ h)+3\ \cos(4\ h+2\ t\ h)$$
$$-\cos(5\ h+2\ t\ h)-2\ \cos(h+2\ t\ h)$$
$$+\cos(-6\ h+2\ t\ h)+2\ \cos(-2\ h+t\ h)$$
$$-\cos(t\ h-8\ h)+2\ \cos(t\ h+8\ h)$$
$$-2\ \cos(-5\ h+2\ t\ h)+\cos(t\ h-7\ h)$$
$$+3\ \cos(t\,h)-2\ \cos(2\,t\,h)+\cos(7\,h+2\,t\,h)$$
$$-\cos\,(t\ h+9\ h))/(\sin^4\,(h)/\cos\,(h)/(4\ \cos^5(h)$$
$$-8\ \cos^4(h)+\cos^3(h)+5\ \cos^2(h)-\cos(h)$$
$$-1)),$$

$$w^s_{j+2}\big(x_j+t\,h\big)=\ 1/128\ \ (-3\ \cos(h)$$
$$+2\ \cos(2\ h))$$
$$+2\ \cos(3\ h)+\cos(7\ h)-2\ \cos(6\ h)$$
$$+2\ \cos(t\ h-5\ h)+\cos(t\ h+5\ h)$$
$$-3\ \cos(t\ h-3\ h)-\cos(t\ h+7\ h)$$
$$-\cos(t\ h-h)$$
$$+2\ \cos(t\ h+h)+\cos(t\ h-4\ h)$$
$$-3\ \cos(t\ h+4\ h)$$
$$-3\ \cos(-3\ h+2\ t\ h)+4\ \cos(3\ h+2\ t\ h)$$
$$+4\ \cos(-h+2\ t\ h)+\cos(-2\ h+2\ t\ h)$$
$$+\cos(6\ h+2\ t\ h)-2\ \cos(2\ h+2\ t\ h)$$
$$-\cos(2\ h+t\ h)-\cos(t\ h-6\ h)$$
$$+2\ \cos(t\ h+6\ h)$$
$$+\cos(-4\ h+2\ t\ h)+\cos(4\ h+2\ t\ h)$$
$$-3\ \cos(5\ h+2\ t\ h)-2\ \cos(h+2\ t\ h)$$
$$+2\ \cos(t\ h)$$
$$-2\ \cos(2\ t\ h))/(\sin^4(h)cos(h)(4\ \cos^5\,(h)$$
$$-8\ \cos^4(h)+\cos^3(h)$$
$$+5\ \cos^2(h)-\cos(h)-1)),$$

$$w^s_{j-1}\big(x_j+t\,h\big)=\ 1/128\ \ (1-\cos(h)+\cos(2\ h)$$
$$-2\ \cos(4\ h))+\cos\,(3\ h)+2\ \cos\,(7\ h)$$
$$-\cos\,(5\ h)$$
$$-\cos(9\ h)-\cos(6\ h)+\cos(8\ h)$$
$$-2\ \cos(t\ h-5\ h)$$
$$-\cos(t\ h+7\ h)+\cos(t\ h-4\ h)$$
$$+3\ \cos(t\ h+4\ h)$$
$$-3\ \cos(3\ h+2\ t\ h)+2\ \cos(-h+2\ t\ h)$$
$$-\cos(-2\ h+2\ t\ h)-\cos(6\ h+2\ t\ h)$$
$$-2\ \cos(2\ h+t\ h)+2\ \cos(t\ h-6\ h)$$
$$-2\ \cos(t\ h+6\ h)-3\ \cos(-4\ h+2\ t\ h)$$
$$+\cos(4\ h+2\ t\ h)+2\ \cos(5\ h+2\ t\ h)$$
$$-\cos(h+2\ t\ h)+\cos(t\ h-9\ h)$$
$$+2\ \cos(t\ h+3\ h)$$
$$+2\ \cos(-6\ h+2\ t\ h)+2\ \cos(-2\ h+t\ h)$$
$$-2\ \cos(t\,h-8\,h)+\cos(t\,h+8\ h)$$
$$+\cos(-5\,h+2\,t\,h)$$
$$-\cos(-7\,h+2\,t\,h)-3\cos(t\,h)$$
$$+2\cos(2\,t\,h))/(\sin^4\,(h)cos(h)(4\cos^5\,(h)$$
$$-8\cos^4(h)+\cos^3(h)$$
$$+5\cos^2(h)-\cos(h)-1)),$$

$$w^s_{j-2}\big(x_j+t\,h\big)=\ 1/128\ \ (-3\ \cos(h)$$
$$+2\ \cos(2\,h))$$
$$+2\ \cos(3\ h)+\cos(7\ h)-2\ \cos(6\ h)$$
$$+\cos(t\,h-5\,h)+2\ \cos(t\ h+5\,h)$$
$$+2\ \cos(t\,h-h)$$
$$-\cos(t\,h+h)-3\ \cos(t\,h-4\ h)$$
$$+\cos(t\,h+4\ h)$$
$$+4\ \cos(-3\ h+2\ t\ h)-3\ \cos(3\ h+2\,t\,h)$$
$$-2\ \cos(-h+2\,t\,h)-2\ \cos(-2\,h+2\,t\,h)$$
$$+\cos(2\,h+2\,t\,h)+2\ \cos(t\,h-6\ h)$$
$$-\cos(t\ h+6\ h)+\cos(-4\ h+2\,t\,h)$$
$$+\cos(4\ h+2\ t\,h)+4\ \cos(h+2\,t\,h)$$
$$-3\ \cos(t\ h+3\ h)+\cos(-6\ h+2\,t\,h)$$
$$-\cos(-2\ h+t\,h)-3\ \cos(-5\ h+2\,t\,h)$$
$$-\cos(t\,h-7\,h)+2\ \cos(t\,h)$$
$$-2\cos(2t\,h))/(\sin^4(h)cos(h)/(4\cos^5\,(h)$$
$$-8\ cos^4(h)+\cos^3(h)$$
$$+5\cos^2(h)-\cos(h)-1)).$$

It is easy to calculate that on the interval $[x_j,x_{j+1}]$, the middle trigonometric basis functions satisfy the inequalities:

$\left|w^s_j\right|\le 1,\quad |w^s_{j+1}|\le 1,\quad |w^s_{j-1}|\le 0.21,\ |w^s_{j-2}|\le 0.08,\ |w^s_{j+2}|\le 0.115.$

We construct an approximation with the middle splines on the interval $\left[x_j,x_{j+1}\right]$ according to the formula:

$$u^j_{S4}(x)=\sum_{i=j-2}^{j+2}u(x_i)w^s_i(x),\ x\in\left[x_j,x_{j+1}\right],\quad (5)$$

In author [23]'s paper, a technique for constructing the error of approximation of functions by nonpolynomial splines is given. In the case of using local trigonometric splines of the fifth order of approximation, following this technique, we note the following. According to this theory, on the interval $[x_j, x_{j+1}]$, we represent the function $u(x)$ as

$$u(x) = \frac{2}{3}\int_{x_j}^{x}\left(5u^{(3)} + u^{(5)} + 4u'\right)sin^4\left(\frac{x}{2} - \frac{t}{2}\right)dt$$

$$+c_1\sin(x) + c_2\cos(x) + c_3\sin(2x) + c_4\cos(2x) + c_5.$$

Here $c_i, i = 1,2,3,4,5$, are arbitrary constants.

It is easy to see that the expression $5u^{(3)} + u^{(5)} + 4u' = 1$ when

$$u(x) = c_1\sin(x) - c_2\cos(x) + \left(\frac{1}{2}\right)c_3\sin(2x) - \frac{1}{2}c_4\cos(2x) + \frac{x}{4} + c_5.$$

And it is easy to see that the expression

$$5u^{(3)} + u^{(5)} + 4u' = 0,$$

when

$$u(x) = c_1\sin(x) + c_2\cos(x) + c_3\sin(2x) + c_4\cos(2x) + c_5.$$

It can be obtained that the next inequality is valid:

$$\left|u_{S4}^{j}(x) - u(x)\right| \le 0.0243\, h^5||5u^{(3)} + u^{(5)} + 4u'||.$$

Having solved the systems of equations (2), (4) we obtain the formulas of the left trigonometric basis splines. Having solved the systems of equations (3), (4) we obtain the formulas of the right trigonometric basis splines.

We construct the approximation with the left splines on the interval $[x_j, x_{j+1}]$ according to the formula:

$$u_{L4}^{j}(x) = \sum_{i=j}^{j+4} u(x_i)w_i^L(x),\ x \in [x_j, x_{j+1}]. \quad (6)$$

We construct the approximation with the right splines on the interval $[x_j, x_{j+1}]$ according to the formula:

$$u_{R4}^{j}(x) = \sum_{i=j-3}^{j+1} u(x_i)w_i^R(x),\ x \in [x_j, x_{j+1}]. \quad (7)$$

Note that approximations with the trigonometric splines have the following properties:

$$u_4^j(x) - u(x) = 0,\ x \in [x_j, x_{j+1}],$$

when $u = 1, \sin(x), \cos(x), \sin(2x), \cos(2x)$.

2.3 Exponential Splines of the Fifth Order of Approximation

In the case of applying a system of functions

$$\varphi_0 = 1, \varphi_1 = \exp(x), \varphi_2 = \exp(-x),$$

$$\varphi_3 = \exp(2x), \varphi_4 = \exp(-2x) \quad (8)$$

we obtain the exponential basis splines.
Let supp $w_j = [x_{j-4}, x_{j+1}]$. Solving a system of equations, we obtain the left exponential basis splines (see (2)) of the form:

$$w_j^L(x) = \left(\exp(x) - \exp(x_{j+1})\right)\left(\exp(x) - \exp(x_{j+3})\right)\left(\exp(x) - \exp(x_{j+4})\right)\left(\exp(x) - \exp(x_{j+2})\right)\exp^2(x_j)\,/((\exp(x_{j+2}) - \exp(x_j))(\exp(x_{j+4}) - \exp(x_j))(\exp(x_{j+3}) - \exp(x_j))(-\exp(x_j) + \exp(x_{j+1}))\exp^2(x));$$

$$w_{j+1}^L(x) = -\left(-\exp(x_j) + \exp(x)\right)\left(\exp(x) - \exp(x_{j+3})\right)\left(\exp(x) - \exp(x_{j+4})\right)\left(\exp(x) - \exp(x_{j+2})\right) exp^2(x_{j+1})/((\exp(x_{j+2}) - \exp(x_{j+1}))/(\exp(x_{j+4}) - \exp(x_{j+1}))(\exp(x_{j+3}) - \exp(x_{j+1}))(-\exp(x_j) + exp(x_{j+1}))\exp^2(x));$$

$$w_{j+2}^L(x) = \left(-\exp(x_j) + \exp(x)\right)\left(\exp(x) - \exp(x_{j+1})\right)\left(\exp(x) - \exp(x_{j+3})\right)\left(\exp(x) - \exp(x_{j+4})\right)\exp^2(x_{j+2})/((\exp(x_{j+2}) - \exp(x_{j+4}))(\exp(x_{j+2}) - \exp(x_{j+3}))(\exp(x_{j+2}) - \exp(x_{j+1}))/((\exp(x_{j+2}) - \exp(x_j))\exp^2(x));$$

$$w_{j+3}^L(x) = ((\exp(x_j) - \exp(x))(\exp(x) - \exp(x_{j+1}))\left(\exp(x) - \exp(x_{j+4})\right)\left(-\exp(x) + \exp(x_{j+2})\right)\exp^2(x_{j+3})/((\exp(x_{j+2}) - \exp(x_{j+3}))(\exp(x_{j+4})$$

$$-\exp(x_{j+3}))(\exp(x_{j+3})-\exp(x_{j+1}))(\exp(x_{j+3}) -\exp(x_j))/\exp^2(x));$$

$$w_{j+4}^L(x)=(-\exp(x_j)+\exp(x))(\exp(x)-\exp(x_{j+1}))(\exp(x)-\exp(x_{j+3}))(-\exp(x)+\exp(x_{j+2}))\exp^2(x_{j+4})/((\exp(x_{j+2})-\exp(x_{j+4}))(\exp(x_{j+4})-\exp(x_{j+3}))(\exp(x_{j+4})-\exp(x_{j+1}))(\exp(x_{j+4})-\exp(x_j))/\exp^2(x)).$$

It is easy to calculate that on the interval $[x_j, x_{j+1}]$, the left basis functions satisfy the inequalities: $|w_j^L|\le 1$, $|w_{j+1}^L|\le 1.37$, $|w_{j+2}^L|\le 0.85$, $|w_{j+3}^L|\le 0.24$, $|w_{j+4}^L|\le 0.02$.
We construct an approximation by such splines on the interval $[x_j, x_{j+1}]$ according to the formula:

$$u_{L4}^j(x)=\sum_{i=j}^{j+4} u_i w_i^L(x),\ x\in[x_j,x_{j+1}].$$

Similarly, we obtain the middle exponential basis splines of the form:

$$w_j^S(x)=(\exp^2(x_j)(\exp(x)-\exp(x_{j+1}))(\exp(x)-\exp(x_{j-1}))(\exp(x)-\exp(x_{j-2}))(\exp(x)-\exp(x_{j+2})))/((-\exp(x_j)+\exp(x_{j+1}))(-\exp(x_j)+\exp(x_{j-1}))(-\exp(x_j)+\exp(x_{j-2}))(-\exp(x_j))+\exp(x_{j+2}))\exp^2(x));$$

$$w_{j+1}^S(x)=(-\exp^2(x_{j+1})(\exp(x)-\exp(x_j))(-\exp(x)+\exp(x_{j-1}))(-\exp(x)+\exp(x_{j-2}))(-\exp(x)+\exp(x_{j+2})))/((-\exp(x_{j+1})+\exp(x_j))(\exp(x_{j-1})-\exp(x_{j+1}))(-\exp(x_{j+1})+\exp(x_{j-2}))(\exp(x_{j+2})-\exp(x_{j+1}))\exp^2(x));$$

$$w_{j+2}^S(x)=(\exp^2(x_{j+2})(\exp(x)-\exp(x_j))(\exp(x)-\exp(x_{j+1}))(\exp(x)-\exp(x_{j-1}))(\exp(x)-\exp(x_{j-2})))/((-\exp(x_j)+\exp(x_{j+2}))(\exp(x_{j+2})-\exp(x_{j+1}))/(\exp(x_{j+2})-\exp(x_{j-1}))(\exp(x_{j+2})-\exp(x_{j-2}))\exp^2(x));$$

$$w_{j-1}^S(x)=(\exp^2(x_{j-1})(\exp(x)-\exp(x_j))(\exp(x)-\exp(x_{j+1}))(\exp(x)-\exp(x_{j-2}))*(\exp(x)-\exp(x_{j+2})))/((-\exp(x_j)+\exp(x_{j-1}))(\exp(x_{j-1})-\exp(x_{j+1}))(\exp(x_{j-2})-\exp(x_{j-1}))\exp(x_{j+2}))-\exp(x_{j-1}))/\exp^2(x));$$

$$w_{j-2}^S(x)=(-\exp^2(x_{j-2})(\exp(x)-\exp(x_j))(\exp(x))-\exp(x_{j+1}))(\exp(x)-\exp(x_{j-1}))(\exp(x)-\exp(x_{j+2})))/(-\exp(x_j)+\exp(x_{j-2}))/(-\exp(x_{j+1})+\exp(x_{j-2}))(\exp(x_{j-2})-\exp(x_{j-1}))(\exp(x_{j+2})-\exp(x_{j-2}))\exp^2(x));$$

It is easy to calculate that on the interval $[x_j, x_{j+1}]$, the middle basis functions satisfy the inequalities $|w_j^S|\le 1$, $|w_{j+1}^S|\le 1$, $|w_{j+2}^S|\le 0.02$, $|w_{j-2}^S|\le 0.0095$, $|w_{j-1}^S|\le 0.125$.
We construct an approximation by such splines on the interval $[t_i, t_{i+1}]$ according to the formula

$$u_{S4}^j(x)=\sum_{i=j-2}^{j+2} u_i w_i^S(x),\ x\in[x_j,x_{j+1}],$$

In author [23]'s paper, a method for constructing an estimate of the approximation error with the exponential splines is given. First of all, we note that the function can be represented in the form:

$$u(x)=12\int_{x_j}^{x}\left(u^{(5)}-5u^{(3)}+4u'\right)\frac{(\exp(x)-\exp(t))^4}{\exp(2t+2x)}dt+c_1\exp(x)+c_2\exp(-x)+c_3\exp(2x)+c_4\exp(-2x)+c_5.$$

Here c_i are arbitrary constants. It is easy to see that the expression
$5u^{(3)}+u^{(5)}+4u'=1$ when $u(x)=c_1\exp(x)-\frac{1}{2}c_2\exp(-2x)+\left(\frac{1}{2}\right)c_3\exp(2x)-c_4\exp(-x)+\frac{x}{4}+c_5$.
It can be obtained that the next inequality is valid:

$$\left|u_{S4}^j(x)-u(x)\right|\le 0.03445\,h^5\,\| u^{(5)}-5u^{(3)}+4u' \|.$$

Having solved the systems of equations (2), (8) we obtain the formulas of the left exponential basis splines. Having solved the systems of equations (3), (8) we obtain the formulas of the right exponential basis splines.
We construct the approximation with the left exponential splines on the interval $[x_j, x_{j+1}]$ according to the formula:

$$u_{L4}^j(x)=\sum_{i=j}^{j+4} u(x_i) w_i^L(x),\ x\in[x_j,x_{j+1}],$$

We construct the approximation with the right exponential splines on the interval $[x_j, x_{j+1}]$ according to the formula:

$$u_{R4}^{j}(x) = \sum_{i=j-3}^{j+1} u(x_i) w_i^R(x),\ x \in [x_j, x_{j+1}].$$

Note that approximations with the exponential splines have the following properties:

$$u_4^j(x) - u(x) = 0,\ x \in [x_j, x_{j+1}],$$

when $u = 1, \exp(x), \exp(-x), \exp(2x), \exp(-2x)$.

2.4 Polynomial Splines of the Fifth Order of Approximation

To construct the polynomial splines of the fifth order of approximation, we take:

$$\varphi_0 = 1, \varphi_1 = x, \varphi_2 = x^2, \varphi_3 = x^3, \varphi_4 = x^4.$$

The construction of polynomial basic splines is considered in detail in the author's articles earlier. The middle basis splines are used in the middle of the interval $[a, b]$. The approximation with the middle basis splines has the form (see paper [23]):

$$U_{S4}^{i}(x) = \sum_{j=i-2}^{i+2} u_j w_j^s(x),\ x \in [x_i, x_{i+1}],$$

where

$$w_{i-2}^s(x) = \frac{(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2})}{(x_{i-2} - x_{i-1})(x_{i-2} - x_i)(x_{i-2} - x_{i+1})(x_{i-2} - x_{i+2})},$$

$$w_{i-1}^s(x) = \frac{(x - x_{i-2})(x - x_i)(x - x_{i+1})(x - x_{i+2})}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-1} - x_{i+1})(x_{i-1} - x_{i+2})},$$

$$w_i^s(x) = \frac{(x - x_{i-2})(x - x_{i-1})(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i-2})(x_i - x_{i-1})(x_i - x_{i+1})(x_i - x_{i+2})},$$

$$w_{i+1}^s(x) = \frac{(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_{i-2})(x_{i+1} - x_{i-1})(x_{i+1} - x_i)(x_i - x_{i+2})},$$

$$w_{i+2}^s(x) = \frac{(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})}{(x_{i+2} - x_{i-2})(x_{i+2} - x_{i-1})(x_{i+2} - x_i)(x_{i+2} - x_{i+1})}.$$

It is easy to calculate that on the interval $[x_j, x_{j+1}]$, the middle basis functions satisfy the inequalities:

$$|w_i^s| \le 1,\ |w_{i+1}^s| \le 1,\ |w_{i-1}^s| \le 0.21,$$

$$|w_{i-2}^s| \le 0.08,\ |w_{i+2}^s| \le 0.115.$$

The error of approximation can be written in the form:

$$\left|u(x) - U_{S4}^{i}(x)\right| \le 1.42\, \frac{h^5}{5!} \max_{\tau \in [x_{i-2}, x_{i+2}]} \left|u^{(5)}(\tau)\right| .$$

At the end of the interval $[a, b]$, we apply the approximation with the right splines:

$$U_{R4}^{i}(x) = \sum_{j=i-3}^{i+1} u_j w_j(x),\ x \in [x_i, x_{i+1}],$$

where the basis splines are the following:

$$w_{i-3}(x) = \frac{(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})}{(x_{i-3} - x_{i-2})(x_{i-3} - x_{i-1})(x_{i-3} - x_i)(x_{i-3} - x_{i+1})},$$

$$w_{i-2}(x) = \frac{(x - x_{i-3})(x - x_{i-1})(x - x_i)(x - x_{i+1})}{(x_{i-2} - x_{i-3})(x_{i-2} - x_{i-1})(x_{i-2} - x_i)(x_{i-2} - x_{i+1})},$$

$$w_{i-1}(x) = \frac{(x - x_{i-3})(x - x_{i-2})(x - x_i)(x - x_{i+1})}{(x_{i-1} - x_{i-3})(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-1} - x_{i+1})},$$

$$w_i(x) = \frac{(x - x_{i-3})(x - x_{i-2})(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-3})(x_i - x_{i-2})(x_i - x_{i-1})(x_i - x_{i+1})},$$

$$w_{i+1}(x) = \frac{(x - x_{i-3})(x - x_{i-2})(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-3})(x_{i+1} - x_{i-2})(x_{i+1} - x_{i-1})(x_{i+1} - x_i)}.$$

The error of approximation with the right splines can be written in the form:

$$\left|u(x) - U_{R4}^{i}(x)\right| \le 3.63 \frac{h^5}{5!} \max_{\tau \in [x_i, x_{i+4}]} \left|u^{(5)}(\tau)\right| .$$

Note that approximations with the polynomial splines have the following properties:

$$u_4^j(x) - u(x) = 0,\ x \in [x_j, x_{j+1}],$$

when $u = 1, x, x^2, x^3, x^4$.

Consider now the approximation by middle splines on a finite interval $[a, b]$.

It follows from formulae (5), (6), (7) that when approximating with the splines of the fifth order of approximation on a finite interval $[a,b]$, the values of the function are required at points that go beyond this finite interval $[a,b]$. In particular, when approximating with the middle splines on the finite interval $[a,b]$, it is necessary to take into account the values of the function in two additional nodes to the right and to the left of the boundaries of the interval $[a,b]$.

Let us take $[a,b]=[-1,1]$, $h=0.1$. Denote $a_1=a-x_{-1}-x_{-2}$, $b_1=b+x_{n+1}+x_{n+2}$.

$$R=\max_{[a_1,b_1]}\left|U_{S4}^{i}(x)-u(x)\right|.$$

Thus, the grid of knots was extended to the left of the interval $[a,b]$ by two nodes: t_{-1},t_{-2} and to the right of the interval $[a,b]$ by two nodes: t_{n+1},t_{n+2}. It was assumed that the function values at these additional nodes are known. To calculate the maximum error, each grid interval $[x_j,x_{j+1}]$ was divided into 100 parts. At each division point, an approximation with the cubic splines of the function u was calculated (the calculations were done in Maple, $Digits=15$). Table 1 shows the maxima in absolute value of the actual errors of the approximation with the middle trigonometric splines of functions and their first derivative: Table 2 shows the maxima in absolute value of the actual errors of the approximation with the middle exponential splines of functions and their first derivative. Table 3 shows the maxima in absolute value of the actual errors of the approximation with the middle polynomial splines of functions and their first derivative:

Table 1 The actual errors of the approximation of functions with trigonometric splines

$u(x)$	Approximation of $u(x),u'(x)$	
	Approximation of $u(x)$	Approximation of $u'(x)$
$\cos(x)-\sin(x)$	$0.121\cdot10^{-9}$	$0.121\cdot10^{-10}$
x^5	0.0000442	0.0191
$\sin(5x)$	0.000290	0.0120
$\exp(3x)$	0.000531	0.0305

Table 2 The actual errors of the approximation of functions with the middle exponential splines

$u(x)$	Approximation of $u(x),u'(x)$	
	Approximation of $u(x)$	Approximation of $u'(x)$
$\cos(x)-\sin(x)$	$0.166\cdot10^{-5}$	$0.690\cdot10^{-4}$
x^5	0.0000140	0.000582
$\sin(5x)$	0.000429	0.0178
$\exp(3x)$	0.000161	0.00927

Table 3 The actual errors of the approximation of functions with the middle polynomial splines

$u(x)$	Approximation of $u(x),u'(x)$	
	Approximation of $u(x)$	Approximation of $u'(x)$
$\cos(x)-\sin(x)$	$0.167\cdot10^{-6}$	$0.694\cdot10^{-5}$
x^5	0.0000142	0.000590
$\sin(5x)$	0.000358	0.0148
$\exp(3x)$	0.000329	0.0189

Table 4 shows the maxima in absolute value of the errors of the theoretical approximation of functions with the trigonometric splines and with the exponential splines

Table 4. The errors of the theoretical approximation of functions with the trigonometric splines and with exponential splines

$u(x)$	Approximation of $u(x)$ with	
	trigonometric splines	exponential splines
$\cos(x)-\sin(x)$	0.0	0.000004871
x^5	0.000107	0.0000551
$\sin(5x)$	0.000612	0.00130
$\exp(3x)$	0.00190	0.000830

It is easy to see that the results presented in the Tables confirm the theoretical estimates. In the next section, we apply exponential, trigonometric and polynomial splines to solve integro-differential equations.

3 Problem Solution

In this section, three examples are given to illustrate the application of the splines in the solving of integro-differential equations.

For an approximate calculation of the integral $\int_a^b f(x)dx$, we use the Newton-Cotes rule. Let $m \geq 1$ be integer number. As is known, the Newton-Cotes quadrature rules have the form:

$$\int_a^b f(x)dx \approx (b-a)\sum_{k=0}^{m} B_k^m f(a+k\tau),$$

$$\tau = \frac{(b-a)}{m},$$

where

$$B_k^m = \frac{(-1)^{m-k}}{m\,k!\,(m-k)!}\int_0^m s(s-1)\dots(s-k+1)(s-k-1)\dots(s-m)ds$$

When $m = 4$, we have

$$B_0^4 = B_4^4 = \frac{7}{90}, B_1^4 = B_3^4 = \frac{32}{90}, B_2^4 = \frac{12}{90}.$$

For constructing the numerical method of solving the integro-ifferential equation with the splines we can take

$$\int_a^b K(x,s)u(s)ds = \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} K(x,s)u(s)ds \approx \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} K(x,s)\tilde{u}(s)ds = \sum_{j=0}^{n-1} J_j,$$

where $\tilde{u}$ is the approximation of u,

$$J_j = \int_{x_j}^{x_{j+1}} K(x,s)\tilde{u}(s)ds.$$

In the case of using the middle splines we get

$$J_j = \int_{x_j}^{x_{j+1}} K(x,s)\sum_{k=j-2}^{j+2} u_k w_k(s)\,ds.$$

In the case of using the left splines we get

$$J_j = \int_{x_j}^{x_{j+1}} K(x,s)\sum_{k=j}^{j+4} u_k w_k(s)\,ds.$$

Thus, using the Newton-Cotes rule when $m = 4$ we have in case of the left splines:

$$J_j = (\frac{7h}{90})\,K(x,x_j)\sum_{k=j}^{j+4} u_k w_k(x_j) +$$

$$+\frac{32h}{90}\,K(x,x_j+h/4)\sum_{k=j}^{j+4} u_k w_k(x_j+h/4)$$

$$+\frac{12h}{90}\,K(x,x_j+h/2)\sum_{k=j}^{j+4} u_k w_k(x_j+h/2)$$

$$+\frac{32h}{90}\,K(x,x_j+3h/4)\sum_{k=j}^{j+4} u_k w_k(x_j+3h/4)$$

$$+\frac{7h}{90}\,K(x,x_j+h)\sum_{k=j}^{j+4} u_k w_k(x_j+h).$$

Example 1. First let us solve the Volterra integro-differential equation

$$u' + \sin(x) - 1 - \int_0^x K(x,t)u(t)dt = 0,$$

when $0 \leq x \leq 1$, $K(x,t) = 1$, $u(0) = 0$. The exact solution is the next: $u(t) = \sin(x)$.

The calculations were carried out in the Maple environment. Fig.1. shows the plot of the error of approximation of the solution of Example 2 obtained with the trigonometric splines when $n = 8$.

In the figures along with the abscissa axis, the grid nodes from the interval [0,1] are marked with blue circles.

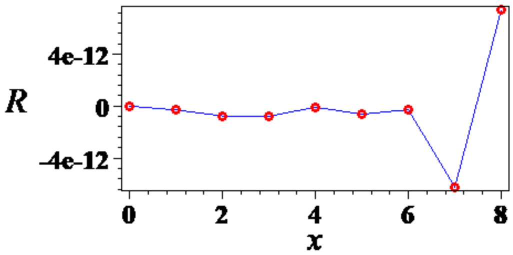

Fig.1. The plot of the error of approximation obtained with the trigonometric splines (Example 1)

Now the goal of the section is to inspect the numerical technique to approximate the solution of the linear second-order Fredholm integro-differential equations (FIDEs) of the form:

$$u''(x) = \alpha(x)u'(x) + \beta(x)u(x) + \gamma(x) + \int_a^b K(x,t)u(t)dt,$$

$a \le x \le b$, with boundary conditions at two points $u(a) = u_0, u(b) = u_n$.

Example 2. Let us apply our theory to solve the integral equation

$$u'' + \sin(x) + x(\sin(1) - \cos(1)) - \int_0^1 K(x,t)u(t)dt = 0$$

$0 \le x \le 1$, with boundary conditions $u(0) = 0, u(1) = \sin(1)$. The exact solution is the next: $u(t) = \sin(x)$.

Fig.2. shows the plot of the error of approximation of the solution of Example 2 obtained with the polynomial splines when $n = 16$

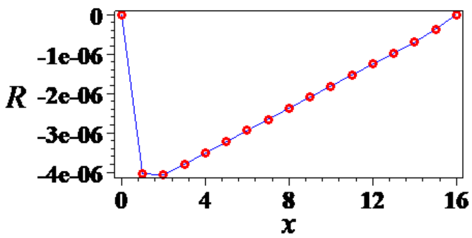

Fig.2. The plot of the error of approximation obtained with the polynomial splines (Example 2)

Fig.3. shows the plot of the error of approximation of the solution of Example 2 obtained with the trigonometric splines when $n = 16$

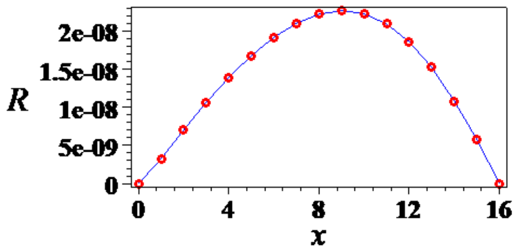

Fig.3. The plot of the error of approximation obtained with the trigonometric splines (Example 2)

The half-sweep (HS) concept is combined with the refinement of the successive over-relaxation (RSOR) iterative method to create the new half-sweep successive over-relaxation (HSRSOR) iterative method, which is implemented to get the numerical solution of a system of linear algebraic equations (see paper [1]). In paper [1] the applicability of the half-sweep successive over-relaxation (HSRSOR) method has been successfully proven.

Example 3. (This example is taken from the paper [1]).

$$u'' - \exp(x) + x - \int_0^1 K(x,t)u(t)dt = 0\,,$$

$K(x,t) = x\,t$. Exact solution is $u(t) = \exp(x)$, $u(0) = 1, u(1) = \exp(1)$.

Method FSRSOR-3LRFD from paper [1] gives the error of approximation 6.0313E-06 when 32 nodes were taken.

Fig.4. shows the plot of the error of approximation of the solution of Example 3 obtained with the exponential splines when $n = 32$, Digits=20. Fig.5. shows the plot of the error of approximation of the solution of Example 3 obtained with the exponential splines when $n = 16$, Digits=20.

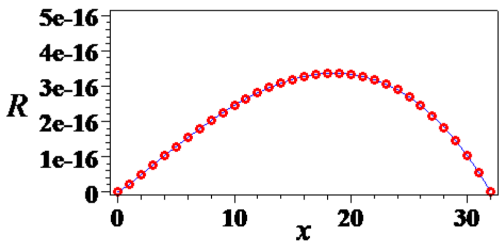

Fig.4. The plot of the error of approximation obtained with the exponential splines, 32 nodes (Example 3)

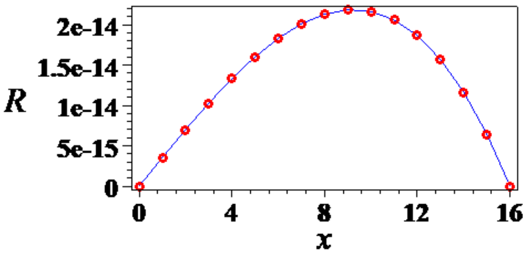

Fig.5. The plot of the error of approximation obtained with the exponential splines, 16 nodes (Example 3)

Fig.6. shows the plot of the error of approximation of the solution of Example 3 obtained with the polynomial splines when $n = 16$, Digits=20

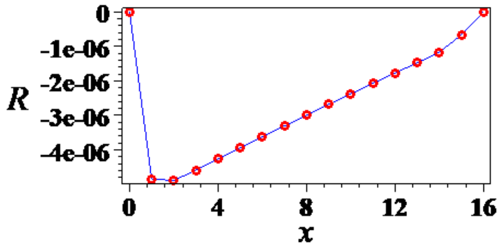

Fig.6. The plot of the error of approximation obtained with the polynomial splines, 16 nodes (Example 3)

Fig.7. shows the plot of the error of approximation of the solution of Example 3 obtained with the trigonometric splines when $n = 16$, Digits=20

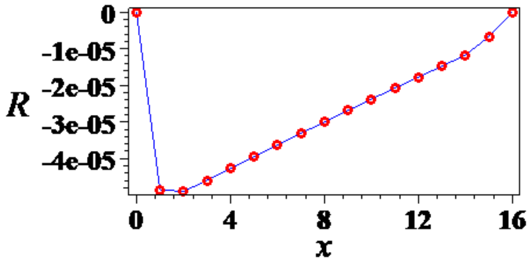

Fig.7. The plot of the error of approximation obtained with the trigonometric splines, 16 nodes (Example 3)

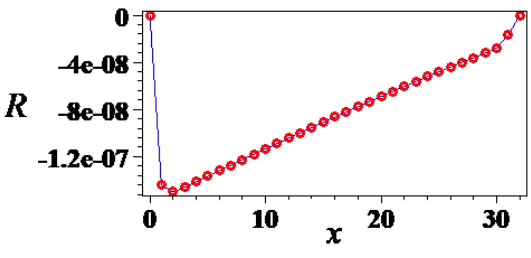

Fig.8. The plot of the error of approximation obtained with the polynomial splines, 32 nodes (Example 3) $R_{max} = 0.146 \cdot 10^{-6}$

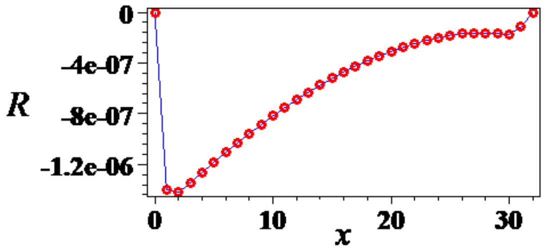

Fig.9. The plot of the error of approximation obtained with the trigonometric splines, 32 nodes (Example 3) $R_{max} = 0.140 \cdot 10^{-5}$

The advantages of using such splines include the ability to determine not only the values of the desired function at the grid nodes, but also the first derivative at the grid nodes. The obtained values can be connected by lines using the splines. Thus, after interpolation, we can obtain the value of the solution at any point of the considered interval. Several numerical examples are given.

From the results presented in the numerical examples, it follows that before the numerical solution of the integral equation, the kernel and the right side of this equation should be analyzed. If the kernel and the right side are a trigonometric expression, then it is advisable to use a numerical method for solving the integral equation based on trigonometric splines.

If the kernel and the right side are an exponential expression, then it is advisable to apply a numerical method for solving the integral equation based on exponential splines. If the kernel and the right side are a polynomial expression, then it is advisable to use a numerical method for solving the integral equation based on polynomial splines.

The considered examples show, that the use of non-polynomial splines can give a smaller solution error even with a small number of grid nodes if the choice of spline approximation corresponds to the form of the kernel and the right side of the integral equation.

4 Conclusion

In this paper, we construct a solution to an integro-differential equation using non-polynomial splines on a uniform grid of nodes. To apply quadrature formulas, the kernel of the integral equation and the solution are assumed to be sufficiently smooth functions. In the future, it is planned to construct a solution to the integro-differential equation on a non-uniform adaptive grid of nodes.

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