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SCHRÖDINGER OPERATOR IN A HALF-PLANE WITH THE NEUMANN CONDITION ON THE BOUNDARY AND A SINGULAR δ-POTENTIAL SUPPORTED BY TWO HALF-LINES, AND SYSTEMS OF FUNCTIONAL-DIFFERENCE EQUATIONS

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We study the asymptotics with respect to distance for the eigenfunction of the Schrödinger operator in a half-plane with a singular δ-potential supported by two half-lines. Such an operator occurs in problems of scattering of three one-dimensional quantum particles with point-like pair interaction under some additional restrictions, as well as in problems of wave diffraction in wedge-shaped and cone-shaped domains. Using the Kontorovich–Lebedev representation, the problem of constructing an eigenfunction of an operator reduces to studying a system of homogeneous functional-difference equations with a characteristic (spectral) parameter. We study the properties of solutions of such a system of second-order homogeneous functional-difference equations with a potential from a special class. Depending on the values of the characteristic parameter in the equations, we describe their nontrivial solutions, the eigenfunctions of the equation. The study of these solutions is based on reducing the system to integral equations with a bounded self-adjoint operator, which is a completely continuous perturbation of the matrix Mehler operator. For a perturbed Mehler operator, sufficient conditions are proposed for the existence of a discrete spectrum to the right of the essential spectrum. Conditions for the finiteness of the discrete spectrum are studied. These results are used in the considered problem in the half-plane. The transformation from the Kontorovich–Lebedev representation to the Sommerfeld integral representation is used to construct the asymptotics with respect to the distance for the eigenfunction of the Schrödinger operator under consideration.

Keywords: functional-difference equations, spectrum, perturbed Mehler operator, asymptotics of eigenfunctions

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1. Introduction

The main goals of this paper are to study the problem of constructing the eigenfunction and its asymptotics for the Schrödinger operator of a special type and to consider the relation between this canonical problem and some spectral properties of a system of functional-difference (FD) equations with a meromorphic matrix potential. An important auxiliary fact is the recently discovered possibility of studying the spectral properties of systems of FD equations by reducing them to integral equations with the so-called perturbed Mehler operator. In addition, our study is motivated by numerous applications, some of which are discussed below.

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We note the interest in the Schrödinger operators with a singular δ-potential supported by surfaces in $\mathbb{R}^d$ (see [1]–[3] and the references therein), as is manifested in the possibility of efficient qualitative study of the spectra of the corresponding self-adjoint operators. The general approaches of spectral theory [4], [5] are used, and the corresponding self-adjoint operators are traditionally defined by semibounded sesquilinear closed forms. However, if the surface of the potential support (for example, a circular cone or a wedge) admits an incomplete separation of variables, then the eigenfunction and eigenvalue problem can be reduced to studying the spectral properties of FD equations [6]–[8]. A remarkable fact is that it is then possible not only to describe the spectrum qualitatively but also to obtain efficient integral representations for the eigenfunctions and to describe the asymptotics of eigenfunctions at long distances.

In [6], the eigenfunctions and their asymptotic behavior at long distances are studied for the Laplace operator with a singular potential supported by a conical surface in three-dimensional space. In the framework of incomplete separation of variables, an integral Kontorovich–Lebedev-type representation for the eigenfunctions is obtained in terms of the solution of an auxiliary FD equation with a meromorphic potential. Solutions of the FD equation are studied by reducing it to an integral equation with a bounded self-adjoint integral operator. This integral operator is a perturbation of the scalar Mehler operator.

In [7], we studied the eigenfunctions that describe the natural oscillations of acoustic waves in angular domains with "semitransparent" boundary conditions. For some values of the spectral parameter in the boundary value problem, we studied the essential and discrete spectra of the equation and described the properties of the corresponding solutions. The study is based on the reduction of FD equations to integral equations of Mehler type with a symmetric kernel.

In contrast to [6], [7], where scalar FD equations are considered, to construct eigenfunctions in this paper, we study a system of two FD equations with a meromorphic potential, and the class of considered matrix potentials is motivated by various applications. One such application is discussed in detail in this paper. We note that it is not difficult to generalize our approach to the case of systems of FD equations of an arbitrary dimension.

The properties of solutions of FD equations have been studied in recent decades along various avenues, and this study was related to the development of new approaches and methods in this field, as well as to numerous applications. Among the most interesting and promising ones, we mention the monodromization method [9], [10], which was used, for example, to describe the geometry of the spectrum of the Harper FD equation. It is also interesting to consider the semiclassical constructions related to difference equations (see [11] and the references therein). Among numerous applications, we point out the problems of (acoustic or electromagnetic) wave scattering in wedge-shaped or cone-shaped domains [12]–[18]. In connection with this work, we note the use of various integral representations (Kontorovich–Lebedev, Sommerfeld–Maliuzhinets, Mellin, Fourier), leading to various FD equations with meromorphic coefficients. Waves on the water surface in a coastal wedge are studied in [19], [20] by the reduction to functional equations of Maliuzhinets type [9]. As an efficient research tool, the FD equations are also known in quantum theory, in particular, when studying the processes of particle decay, many-particle quantum scattering, etc. [21]–[24].

In Sec. 2, we discuss a basic example related to the self-adjoint Schrödinger operator $A_{\rm s}$ with a singular potential and leading to the study of a system of FD equations. The results in Secs. 3–6 are used to construct the eigenfunctions of $A_{\rm s}$ and to study their behavior at long distances. For this, in particular, the integral representations for solutions in the form of Kontorovich–Lebedev and Sommerfeld integrals are used. It is useful to note that the operator $A_{\rm s}$ can be related to the quantum mechanical problem of scattering of three neutral particles with pair interaction given by a δ-function moving along a straight line with some additional restrictions on the character of their interaction. A similar operator also arises in the problem of wave diffraction on a system of semitransparent screens in acoustics or electrodynamics.

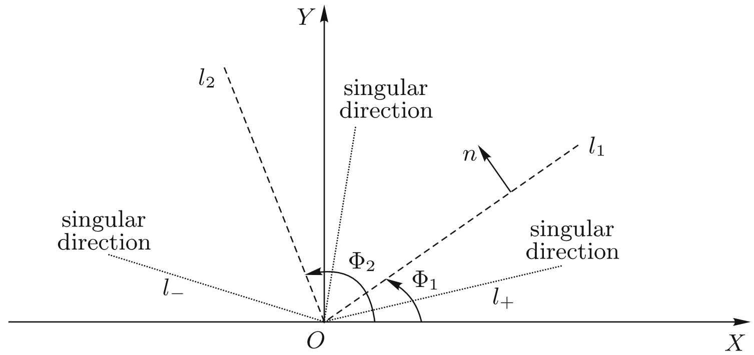

Fig. 1. Domain and support of the potential in $\mathbb{R}^2_+$.

We consider the formulation of the problem on the spectrum and eigenfunctions (including generalized ones) for a system of two FD equations and describe the class of potentials under consideration. The system is reduced to integral equations on a closed interval. The relevant matrix integral operator $\mathbf{K}$ is bounded and self-adjoint but not completely continuous. We represent this operator as the sum of the so-called matrix Mehler operator $\mathbf{M}$ and its compact perturbation $\mathbf{V}$. Thus, the study of the spectral properties of a system of two FD equations is reduced to describing the spectral properties of a matrix integral operator.

In Appendix A, we investigate the simplest matrix analogue $\mathbf{M}$ of the scalar self-adjoint Mehler operator M, discuss its spectrum and eigenfunctions, and calculate the resolvent and the partition of unity. The properties of the kernel of the resolvent are studied and the estimates necessary for the further analysis are obtained. (The results directly follow from Mehler's classic formulas of 1881.) These auxiliary results are used below to study systems of FD equations with a characteristic parameter.

In Sec. 4, we show that the essential spectrum of the perturbed Mehler operator $\mathbf{K}$ coincides with the interval $[0, 1]$. The operator $\mathbf{K}$ is positive, and hence the discrete component of the spectrum can be located to the right of unity. Sufficient conditions for the existence of a discrete component of the spectrum of the operator are written in terms of the potential. With the help of the Birman–Schwinger principle, a criterion for the finiteness of the discrete spectrum is proposed, which is similar to the criterion obtained in [25] for the perturbation of the Carleman operator by Hankel operators.

The results concerning the spectrum of $\mathbf{K}$ are used to describe the spectral properties of FD equations. We introduce the concept of a set of characteristic numbers of a system of equations and of the essential characteristic set. Estimates are obtained that describe the behavior of meromorphic solutions ("eigenfunctions") at infinity along the imaginary axis.

Finally, we construct efficient formulas for the asymptotics of eigenfunctions. For this, we use the Sommerfeld integral representation and traditional asymptotic methods such as the saddle-point method and its "uniform" version.

2. Problem of constructing eigenfunctions of the Schrödinger operator with a singular potential

2.1. Problem statement. We define a self-adjoint Schrödinger operator $A_{\rm s}$, which is considered as an example, by its sesquilinear form a_s in $L_2(\mathbb{R}^2_+)$ that is semibounded, densely defined, and closed.[1] We consider the partition of the half-plane $\Omega = \mathbb{R}^2_+$ ($x = (X, Y) \in \Omega$) into three parts Ω_j, $j = 1, 2, 3$, by

[1]We recall that under certain conditions, as was mentioned in Sec. 1, this operator can be related to the problem of quantum scattering of three one-dimensional particles with point-like pair interaction.

the half-lines l_1 and l_2 (see Fig. 1). We introduce polar coordinates $X = r\cos\varphi$, $Y = r\sin\varphi$ and

$$\Omega_1 = \{(r,\varphi)\colon r>0, 0<\varphi<\Phi_1\},$$
$$\Omega_2 = \{(r,\varphi)\colon r>0, \Phi_1<\varphi<\Phi_2\},$$
$$\Omega_3 = \{(r,\varphi)\colon r>0, \Phi_2<\varphi<\pi\},$$

where $0<\Phi_1<\Phi_2<\pi$. The corresponding quadratic form becomes

$$a_s[U,U] = \int_\Omega \nabla U\cdot\overline{\nabla U}\,dx - \gamma_1\int_{l_1}|U|^2\,ds - \gamma_2\int_{l_2}|U|^2\,ds,$$

where $\gamma_1>0$ and $\gamma_2>0$ are Robin parameters, $\mathrm{Dom}[a_s] = H^1(\Omega)$, $\Omega = \Omega_1\cup l_1\cup\Omega_2\cup l_2\cup\Omega_3$.

The operator A_s generated by the form a_s is self-adjoint and semibounded. It is realized as a Laplacian with a singular δ-potential supported by $l = l_1\cup l_2$, i.e., $A_\mathrm{s} = -\triangle - \gamma\delta_l(x)$ [1]. In what follows, we use its classical realization in terms of equations and boundary conditions. We study the equation

$$A_\mathrm{s}U = EU,$$

where the spectral parameter E is assumed to be negative, $E<0$. It is well known that the essential spectrum $\sigma_\mathrm{e}(A_\mathrm{s})$ of the operator A_s coincides with $[-\gamma^2/4,\infty)$, where $\gamma = \max\{\gamma_1,\gamma_2\}$ (we further assume that $\gamma_1\geqslant\gamma_2$). The spectrum also has a discrete part $\sigma_\mathrm{d}(A_\mathrm{s})$. Our goal is to propose integral representations for eigenfunctions of the discrete spectrum $\sigma_\mathrm{d}(A_\mathrm{s})$ of A_s and to illustrate the use of the results in Secs. 3–6, in particular, to find the values $\sigma_\mathrm{d}(A_\mathrm{s})$ and to verify that the discrete spectrum is nonempty under certain conditions. We formulate these conditions in terms of the potential of FD equations related to the considered example.

2.2. Classical statement and main results. We seek the classical solution $u = u_j$ in Ω_j, $j=1,2,3$, satisfying the equations ($E<0$)

$$\begin{aligned}
-\triangle u_1(r,\varphi) - Eu_1(r,\varphi) &= 0, \qquad (r,\varphi)\in\Omega_1,\\
-\triangle u_2(r,\varphi) - Eu_2(r,\varphi) &= 0, \qquad (r,\varphi)\in\Omega_2,\\
-\triangle u_3(r,\varphi) - Eu_3(r,\varphi) &= 0, \qquad (r,\varphi)\in\Omega_3,
\end{aligned}\tag{1}$$

and the boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{Y=0} = 0,\tag{2}$$

$$\begin{aligned}
\left.\frac{\partial u_1}{\partial n}\right|_{l_1} - \left.\frac{\partial u_2}{\partial n}\right|_{l_1} &= \gamma_1 u_1|_{l_1}, \qquad u_1|_{l_1} = u_2|_{l_1},\\
\left.\frac{\partial u_2}{\partial n}\right|_{l_2} - \left.\frac{\partial u_3}{\partial n}\right|_{l_2} &= \gamma_2 u_2|_{l_1}, \qquad u_2|_{l_2} = u_3|_{l_2},
\end{aligned}\tag{3}$$

where the unit normal n is directed counterclockwise. The condition $u\in H^1(\Omega)$ implies that

$$u(r,\varphi) = C + O(r^{\delta_*}), \qquad \delta_*>0,\ r\to 0,\tag{4}$$

uniformly in φ. We assume that nontrivial solutions (eigenfunctions) of problem (1)–(4) decrease exponentially as $r\to\infty$ and the integral

$$\int_\Omega |u(r,\varphi)|^2 e^{2dr} r\,dr\,d\varphi < \infty\tag{5}$$

is bounded for some positive d. We note that such solutions satisfying (1)–(4) exist for some $E<-\gamma^2/4$, but if $E\geqslant-\gamma^2/4$, then the corresponding nontrivial solutions (those on the continuous spectrum) violate condition (5).

The main new results that we discuss in detail in this paper consist in the following. First, we study sufficient conditions for the existence of the discrete component of the spectrum of the operator A_{s}, whose analysis via the Kontorovich–Lebedev integrals reduces to the description of characteristic values $\Lambda = \Lambda_m$ of the corresponding FD equations of the form $\mathbf{h}(\nu+1) - \mathbf{h}(\nu-1) - 2i\Lambda\mathbf{W}_*(\nu)\mathbf{h}(\nu) = \mathbf{0}$ related to the problem under consideration (here, $\mathbf{W}_*(\nu)$ is a meromorphic matrix potential defined explicitly in Sec. 2.3 in terms of sine-to-cosine ratios). The FD equation corresponds to the integral equation $\mathbf{r}(x) = \Lambda(\mathbf{Kr})(x)$ with the following operator (which we call a perturbed Mehler operator):

$$(\mathbf{Kr})(x) = \frac{1}{\pi}\int_0^1 \frac{dy}{x+y}\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)}\mathbf{r}(y),$$

where $\mathbf{w}(x) = \mathbf{W}(it)\big|_{t=\frac{1}{\pi}\ln(1/x+\sqrt{1/x^2-1})} > 0$, $t > 0$, and $\mathbf{a}$ is a constant diagonal matrix.

Sufficient conditions for the existence of the discrete spectrum $\mu_m = \Lambda_m^{-1}$ of this operator (and hence also of the spectrum $E_m = -[\gamma_1/2\Lambda_m]^2$ of A_{s}) are described in the following theorem.

Theorem 2.1. *Let $\mathbf{V}$ be a self-adjoint compact operator in the space $\mathcal{H} = L_2((0,1);\mathbb{C}^2)$, and for some n, let the inequality*

$$\frac{1}{\pi}\int_0^1 dx\int_0^1 dy\,\frac{\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)}-\mathbf{a}}{y+x}\mathbf{u}_n(y)\overline{\mathbf{u}_n(x)} > \varepsilon_n \tag{6}$$

be satisfied. Then the perturbed Mehler operator $\mathbf{K} = \mathbf{M} + \mathbf{V}$ has a nontrivial discrete spectrum to the right of $\sigma_{\mathrm{e}}(\mathbf{K}) = [0,1]$ (we can take $\varepsilon_n = 1/n$ in the inequality).

In this theorem, $\mathbf{u}_n(x)$ is a singular Weyl sequence that can be constructed explicitly for the unperturbed operator $\mathbf{M} = \mathbf{K} - \mathbf{V}$ and which corresponds to the end of the essential spectrum $\mu = 1$ of this operator. A simpler sufficient condition is given by inequality (21).

Second, using the Birman–Schwinger principle, we obtain a condition for the finiteness of the discrete component of the spectrum in terms of the potential in the system of FD equations. Namely, the following assertion holds.

Theorem 2.2. *Let $\alpha > 3/2$, let the operator $\mathbf{V} \geq 0$, $\mathbf{V} \in S_2$, be of Hilbert–Schmidt class and, in addition, let the operator $\mathbf{Q}^\alpha\mathbf{V}\mathbf{Q}^\alpha \in S_\infty$ be compact. Then the total number $N(1)$ of eigenvalues of the operator $\mathbf{K} = \mathbf{M} + \mathbf{V}$ that are greater than $\mu = 1$ is finite and satisfies the estimate*

$$N(1) \leq (\|\mathbf{V}\|_2 + G_\alpha\|\mathbf{Q}^\alpha\mathbf{V}\mathbf{Q}^\alpha\|)^2, \tag{7}$$

where G_α is a constant, $\mathbf{Q} := Q\mathbf{I}$, $[Qf](t) := \langle\ln t\rangle f(t)$, and $\langle\ln t\rangle = \ln(2/t)$, $f \in L_2(0,1)$.

If in addition sufficient conditions (6) *or* (23) (*or* (21)) *are satisfied, then the discrete spectrum of* $\mathbf{K}$ *located to the right of $\mu = 1$ is not empty.*

Finally, the asymptotic behavior of the eigenfunction is calculated and its exponential decrease is shown.

Lemma 2.1. *The eigenfunction u_m exponentially decreases as $r \to \infty$ (in accordance with* (44), (45) *outside the singular directions in Ω_1); otherwise, in a neighborhood of singular directions, the asymptotics has form* (46) *(in Ω_1) and depends on a Fresnel-type integral. The asymptotics has a similar structure in the domain $\Omega_{2,3}$.*

This also proves the existence of so-called singular directions near which the character of the exponential decrease of the eigenfunction changes.

2.3. Kontorovich–Lebedev integrals and reduction to FD equations. We seek the classical solution of Eqs. (1) with $0 < \varphi < \pi$ in the form of integrals

$$
\begin{aligned}
u_1(r,\varphi) &= \frac{1}{i\pi}\int_{-i\infty}^{i\infty} \sin(\pi\nu) K_\nu(\kappa r) \frac{\cos(\nu\varphi)}{\cos(\Phi_1\nu)} H_1(\nu)\, d\nu, \qquad \varphi \in [0,\Phi_1],\\
u_2(r,\varphi) &= \frac{1}{i\pi}\int_{-i\infty}^{i\infty} \sin(\pi\nu) K_\nu(\kappa r) \Bigl(\frac{\cos(\nu[\Phi_2-\varphi])}{\cos(\nu[\Phi_2-\Phi_1])} H(\nu) + \\
&\qquad\qquad + \frac{\sin(\nu[\Phi_1-\varphi])}{\sin(\nu[\Phi_1-\Phi_2])}\tilde h(\nu)\Bigr)\, d\nu, \qquad \varphi \in [\Phi_1,\Phi_2],\\
u_3(r,\varphi) &= \frac{1}{i\pi}\int_{-i\infty}^{i\infty} \sin(\pi\nu) K_\nu(\kappa r) \frac{\cos(\nu[\pi-\varphi])}{\cos(\nu[\pi-\Phi_2])} H_3(\nu)\, d\nu, \qquad \varphi \in [\Phi_2,\pi],
\end{aligned}
\tag{8}
$$

where $\kappa = \sqrt{-E}$. Kontorovich–Lebedev representation (8) separates the variables r and φ, and ν is the separation variable. If the integrals converge uniformly and rapidly, then the equations are satisfied, because it can be verified that

$$
(\kappa r)^2 \Bigl\{ \frac{d^2}{d(\kappa r)^2} + \frac{1}{\kappa r}\frac{d}{d\kappa r} - \Bigl(1 + \frac{\nu^2}{(\kappa r)^2}\Bigr)\Bigr\} K_\nu(\kappa r) u_\nu(\varphi) + \Bigl(\frac{d^2}{d\varphi^2} + \nu^2\Bigr) u_\nu(\varphi) K_\nu(\kappa r) = 0,
$$

where $u_\nu(\varphi) = \cos(\nu\varphi)$ or $u_\nu(\varphi) = \sin(\nu\varphi)$. We choose the integrands in (8) so as to satisfy Neumann condition (2). Now we consider boundary conditions (3). First, we proceed formally to formulate sufficient conditions for the unknowns $H_1(\nu)$, $H(\nu)$, $\tilde h(\nu)$, and $H_3(\nu)$, i.e., to describe an appropriate class of functions. The conditions for the continuity of u on $l_{1,2}$ in (3) imply the relations

$$
\begin{aligned}
&H_1(\nu) = H(\nu),\\
&\frac{1}{\cos(\nu[\Phi_2-\Phi_1])} H(\nu) + \tilde h(\nu) = H_3(\nu),
\end{aligned}
\tag{9}
$$

and hence only two functions are independent, for example, $H(\nu)$ and $H_3(\nu)$, while the other functions can be expressed in terms of them, see (9). From Robin-type conditions (3), we have

$$
\begin{aligned}
&\frac{1}{\kappa r}\Bigl(\frac{\partial u_1}{\partial\varphi} - \frac{\partial u_2}{\partial\varphi}\Bigr)\Bigr|_{\varphi=\Phi_1} - \gamma_1 u_1|_{\varphi=\Phi_1} = \\
&= \frac{1}{i\pi}\int_{-i\infty}^{i\infty} d\nu\, \sin(\pi\nu) \Bigl\{ \frac{K_\nu(\kappa r)}{\kappa r}\Bigl[H(\nu)\frac{-\nu\sin(\nu[\Phi_2-\Phi_1])}{\cos(\nu[\Phi_2-\Phi_1])} + \tilde h(\nu)\frac{-\nu}{\sin(\nu[\Phi_2-\Phi_1])} - \\
&\qquad - H(\nu)\nu\tan(\Phi_1\nu)\Bigr] - \frac{\gamma_1}{\kappa} H_1(\nu) K_\nu(\kappa r)\Bigr\} = 0.
\end{aligned}
$$

We use the identity

$$
\frac{K_\nu(z)}{z} = \frac{K_{\nu+1}(z) - K_{\nu-1}(z)}{2\nu}
$$

to find

$$
\begin{aligned}
&\frac{1}{i\pi}\int_{-i\infty}^{i\infty} d\nu\,(-\sin(\pi\nu))\Bigl\{ \frac{K_{\nu+1}(\kappa r) - K_{\nu-1}(\kappa r)}{2}\Bigl[H(\nu)\Bigl(\frac{\sin(\nu[\Phi_2-\Phi_1])}{\cos(\nu[\Phi_2-\Phi_1])} + \tan(\Phi_1\nu)\Bigr) + \\
&\qquad + \tilde h(\nu)\frac{1}{\sin(\nu[\Phi_2-\Phi_1])}\Bigr] + \frac{2\gamma_1}{\kappa} H(\nu)\frac{K_\nu(\kappa r)}{2}\Bigr\} = \\
&= \frac{1}{2i\pi}\int_{-i\infty+1}^{i\infty+1} d\nu\, \sin(\pi\nu) K_\nu(\kappa r) h_1(\nu-1) - \\
&\quad - \frac{1}{2i\pi}\int_{-i\infty-1}^{i\infty-1} d\nu\, \sin(\pi\nu) K_\nu(\kappa r) h_1(\nu+1) - \\
&\quad - \frac{2\gamma_1}{\kappa}\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} d\nu\, \sin(\pi\nu) H(\nu) K_\nu(\kappa r) = 0,
\end{aligned}
$$

where we replaced the integration variables as $\nu \pm 1 \to \nu$ and introduced

$$h_1(\nu) := H(\nu)(\tan(\nu[\Phi_2 - \Phi_1]) + \tan(\Phi_1 \nu)) + \frac{\tilde{h}(\nu)}{\sin(\nu[\Phi_2 - \Phi_1])}.$$

Here, $\tilde{h}(\nu)$ is expressed in terms of $H(\nu)$ and $H_3(\nu)$ due to (9). Deforming the integration contours to the imaginary axis in the first two integrals given above, we obtain

$$\frac{1}{2i\pi} \int_{-i\infty}^{i\infty} d\nu \, \sin(\pi\nu) K_\nu(\kappa r) \left[h_1(\nu - 1) - h_1(\nu + 1) - \frac{2\gamma_1}{\kappa} H(\nu) \right] = 0.$$

As a result, we conclude that if the equation

$$h_1(\nu + 1) - h_1(\nu - 1) + \frac{2\gamma_1}{\kappa} H(\nu) = 0$$

is satisfied, then the Robin condition holds on l_1 in (3). Similarly, from the boundary condition on l_2, taking the above consideration into account, we obtain the system of equations

$$\begin{aligned} h_1(\nu + 1) - h_1(\nu - 1) + \frac{2\gamma_1}{\kappa} H(\nu) = 0, \\ h_2(\nu + 1) - h_2(\nu - 1) + \frac{2\gamma_2}{\kappa} H_3(\nu) = 0, \end{aligned} \tag{10}$$

where

$$h_2(\nu) := H_3(\nu)(\tan(\nu[\pi - \Phi_2]) - \cot(\nu[\Phi_2 - \Phi_1])) + \frac{H(\nu)}{\sin(\nu[\Phi_2 - \Phi_1])}.$$

We write system (10) in terms of an unknown 2-vector $\mathbf{h}(\nu) = (h_1(\nu), h_2(\nu))^{\mathrm{T}}$. We obtain an equation of form (11), which we study in Sec. 3,

$$\mathbf{h}(\nu + 1) - \mathbf{h}(\nu - 1) - 2i\Lambda \mathbf{a} \mathbf{W}(\nu) \mathbf{h}(\nu) = \mathbf{0}, \tag{11}$$

where we introduce the notation $\Lambda = \gamma_1/(2\kappa)$ and $\mathbf{H}(\nu) = (H(\nu), H_3(\nu))^{\mathrm{T}}$,

$$\mathbf{T}(\nu) = \begin{pmatrix} \tan(\nu[\Phi_2 - \Phi_1]) + \tan(\Phi_1\nu) - \dfrac{\sec(\nu[\Phi_2 - \Phi_1])}{\sin(\nu[\Phi_2 - \Phi_1])} & \dfrac{1}{\sin(\nu[\Phi_2 - \Phi_1])} \\ \dfrac{1}{\sin(\nu[\Phi_2 - \Phi_1])} & \tan(\nu[\pi - \Phi_2]) - \cot(\nu[\Phi_2 - \Phi_1]) \end{pmatrix},$$

$$\mathbf{a} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix},$$

$a := \gamma_2/\gamma_1 < 1$, and

$$\mathbf{W}(\nu) = 2i\mathbf{T}^{-1}(\nu) = \frac{2i}{D(\nu)} \times$$
$$\times \begin{pmatrix} \tan(\nu[\pi - \Phi_2]) - \cot(\nu[\Phi_2 - \Phi_1]) & \dfrac{-1}{\sin(\nu[\Phi_2 - \Phi_1])} \\ \dfrac{-1}{\sin(\nu[\Phi_2 - \Phi_1])} & \left(\tan(\nu[\Phi_2 - \Phi_1]) + \tan(\Phi_1\nu) - \dfrac{\sec(\nu[\Phi_2 - \Phi_1])}{\sin(\nu[\Phi_2 - \Phi_1])} \right) \end{pmatrix},$$

where

$$\begin{aligned} D(\nu) := \; & \det \mathbf{T}(\nu) = (\tan(\nu[\Phi_2 - \Phi_1]) + \\ & + \tan(\Phi_1\nu))(\tan(\nu[\pi - \Phi_2]) - \cot(\nu[\Phi_2 - \Phi_1])) - \frac{2\tan(\nu[\pi - \Phi_2])}{\sin(2\nu[\Phi_2 - \Phi_1])}. \end{aligned}$$

We say that a vector $\mathbf{h}(\nu)$ is of class $\mathcal{M}$ of meromorphic vectors if its components $h_j(\nu)$, $j = 1, 2$, satisfy the conditions

- $h_j(\nu) = h_j(-\nu)$;
- h_j is holomorphic in $\Pi_{1+\delta} \setminus \{\pm 1\}$, $\Pi_{1+\delta} = \{\nu \in \mathbb{C}\colon |\mathrm{Re}\,\nu| < 1 + \delta\}$ for some $\delta > 0$;
- $|h_j(\nu)| < \mathrm{const}\,|\exp(i\nu[\pi/2 + \delta_0])|$, $\nu \to i\infty$, $\nu \in \Pi_{1+\delta}$ for $\delta_0 \in (0, \pi/2)$.

We further assume that $\mathbf{h}(\nu)$ is of class $\mathcal{M}$, and hence $\mathcal{W}$ satisfies the estimate $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{O}(e^{\pm i q_* \nu})$ (with $q_* = 2\min\{\Phi_2 - \Phi_1, \Phi_1, \pi - \Phi_2\} > 0$ in our case). The explicit description of the class of potentials $\mathcal{W}$ is discussed in Sec. 3.

Solutions from the class $\mathcal{M}$ ensure the uniform convergence of integrals in (8) due to the asymptotics

$$K_\nu(z) \sim \mathrm{const}\, \frac{\nu^{-1/2}\cos(\nu[\pi/2 + |\arg(z)|])}{\sin(\pi\nu)}$$

as $\nu \to i\infty$ with $\mathrm{Re}(\nu) = 0$, $|\arg z| \leqslant \pi/2$, and a fixed $|z|$. We arrive at the following assertion.

Proposition 2.1. *Assume that* $\mathbf{h}(\,\cdot\,)$ *is of class* $\mathcal{M}$ *and provides a nontrivial solution of Eq.* (11) *for some* Λ. *Then the Kontorovich–Lebedev integrals in* (8) *determine a classical solution of problem* (1)–(3) *for the corresponding* $E = -\gamma_1^2/4\Lambda^2$.

We note that condition (4) can be verified using the Kontorovich–Lebedev representations (see 5.2.2 in [18]). But the verification of estimate (5) is more difficult and follows from the asymptotics for the eigenfunction that we obtain below.

3. FD equations with a characteristic parameter and reduction to integral equations

We study systems of two coupled FD equations (11) for the unknown vector $\mathbf{h}(\nu) = (h_1(\nu), h_2(\nu))^{\mathrm{T}} \in \mathcal{M}$; Λ is a characteristic (in general, complex) parameter,[2] and $\mathbf{W}(\,\cdot\,)$ is a potential, that is, a matrix with meromorphic coefficients.

It turns out that under certain conditions on the potential $\mathbf{W}$, the equation has nontrivial solutions from the above class, and these solutions exist only for some real values of Λ that form a discrete set C_{d}, finite or infinite. By definition, this discrete set is called the set of characteristic values Λ_m, $m = 1, 2, \ldots$, of Eq. (11). The corresponding solutions from $\mathcal{M}$ are called the vector eigenfunctions $\mathbf{h}_m$ of the equation. In what follows, we also define the set of essential values C_{e} of the characteristic parameter Λ. But the focus of our attention is on the set of characteristic values C_{d}. The description of the set of characteristic values C_{d} and the essential characteristic set C_{e} for Eq. (11), as well as of the corresponding solutions, is called the description of the spectral properties of Eq. (11). It turns out that these sets are naturally and canonically related to the spectrum of some self-adjoint integral operator $\mathbf{K}$.

We consider the class $\mathcal{W}$ of meromorphic potentials $\mathbf{W}(\,\cdot\,)$ such that

- $\mathbf{W}(\nu) = -\mathbf{W}(-\nu)$ is odd;
- $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{o}(1)$ as $\nu \to \pm i\infty$ along the imaginary axis;
- $\mathbf{W}(\nu) \sim \mathbf{C}\nu^{-1}$ as $\nu \to 0$, where $\mathbf{C}$ is a constant matrix;
- $\mathbf{W}(it) > 0$ for $t > 0$.

[2] With applications in mind, it is convenient to separate the constant matrix $\mathbf{a}$ multiplying the matrix potential.

It is significant in what follows that the potential $\mathbf{W}$ is positive on the positive part of the imaginary axis. In applications, the sharper estimate $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{O}(e^{-b|\nu|})$ (or $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{O}(1/\nu)$) holds as $\nu \to i\infty$ along the imaginary axis, $b > 0$. The following lemma is used to obtain integral equations (see the proof in [7]) from FD equations.

Lemma 3.1. *Suppose that $q(\nu)$ is holomorphic for $\nu \in \Pi_\delta$ for $\delta > 0$ and $|q(\nu)| \leqslant c_q e^{-\kappa_* |\nu|}$, $|\nu| \to \infty$, $\kappa_* > 0$ in this strip, the function q is odd, $q(\nu) = -q(-\nu)$, and meromorphic. Then the even solution $s(\,\cdot\,)$ of the equations*

$$s(\nu \pm 1) - s(-\nu \pm 1) = \pm 2iq(\nu)$$

that is regular (holomorphic) in the strip $\nu \in \Pi_{1+\delta} \setminus \{\pm 1\}$ ($\nu = 0$ is a simple pole of q) and exponentially decreases in it as $|\nu| \to \infty$, is given by the expression

$$s(\nu) = -\frac{1}{2} \int_{-i\infty}^{i\infty} d\tau\, q(\tau) \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu}, \qquad \nu \in \Pi_{1+\delta},$$

and $s(\nu)$ can be extended as a meromorphic function by means of FD equations.

Applying Lemma 3.1 to Eq. (11) and taking the properties of $\mathbf{W}$ and $\mathbf{h}$ into account, we obtain

$$\mathbf{h}(\nu) = -\frac{\Lambda}{2} \int_{-i\infty}^{i\infty} d\tau \, \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu} \mathbf{a}\mathbf{W}(\tau)\mathbf{h}(\tau), \qquad \nu \in \Pi_{1+\delta}. \tag{12}$$

Because the integrand in (12) is even, we reduce the integration to the semiaxis $[0, i\infty)$, introduce a new unknown vector

$$\mathbf{q}(\nu) = \sqrt{\mathbf{W}(\nu)}\mathbf{h}(\nu), \qquad \mathbf{W}(\nu) > 0, \qquad \nu \in [0, i\infty), \tag{13}$$

and obtain the sought equation

$$\mathbf{q}(\nu) = -\Lambda \int_{0}^{i\infty} d\tau \, \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu} \sqrt{\mathbf{W}(\nu)}\mathbf{a}\sqrt{\mathbf{W}(\tau)}\mathbf{q}(\tau), \tag{14}$$

where $\nu \in [0, i\infty)$.

The procedure of reconstructing the unknown meromorphic vector $\mathbf{h}(\nu)$ from class $\mathcal{M}$ from its value on the semiaxis $[0, i\infty)$ is as follows. Let there be an integrable (and hence continuous) solution $\mathbf{q}(\nu)$ of integral equation (14) on $[0, i\infty)$ for some Λ, which exponentially decreases at infinity. We use the oddness and define $\mathbf{q}(\nu)$ on the entire imaginary axis. Defining $\mathbf{h}(\nu)$ on the imaginary axis by (14), we continue it into the strip Π_δ for some $\delta > 0$. Integral representation (12) allows calculating the values of $\mathbf{h}(\nu)$ in the regularity strip $\Pi_{1+\delta}$. Indeed, calculating $\mathbf{h}(\nu)$ in some neighborhood of the imaginary axis, we see that the integral in the right-hand side of (12) defines a holomorphic function in the strip $\Pi_{1+\delta}$, because the denominator has no zeros in this strip and the integral converges exponentially and uniformly in ν. (Indeed, we shift the contour $i\mathbb{R}$ to the strip Π_δ such that $h(\nu)$ in the left-hand side of (12) is holomorphic in the strip $\Pi_{1+\delta}$.) The vector $\mathbf{h}(\nu)$ can be continued as a meromorphic function to the whole plane $\mathbb{C}$ by means of FD equation (11).

Lemma 3.2. *Let, for some Λ, $\mathbf{q}$ be a solution of integral equation* (14) *that is integrable on $[0, i\infty)$ and exponentially decreases at infinity. Then the corresponding nontrivial solution $\mathbf{h}(\,\cdot\,) \in \mathcal{M}$ of FD equation* (11) *exists.*

Lemma 3.2 shows that nontrivial solutions of integral equation (14) for some characteristic value of the parameter Λ must be studied. For this, it is convenient to transform the equation to some form involving integration over a finite interval. We introduce a new integration variable and a new unknown function as

$$x = \frac{1}{\cos(\pi\nu)}, \qquad y = \frac{1}{\cos(\pi\tau)}, \qquad x \in [0,1],$$
$$\mathbf{r}(x) = \mathbf{q}(it)\big|_{t=\frac{1}{\pi}\ln(1/x+\sqrt{1/x^2-1})}, \qquad t > 0$$

and then obtain the equation

$$\mathbf{r}(x) = \Lambda(\mathbf{K}\mathbf{r})(x), \tag{15}$$

where $\mathbf{K}$ is an integral operator in $L_2([0,1];\mathbb{C}^2)$,

$$(\mathbf{K}\mathbf{r})(x) = \frac{1}{\pi}\int_0^1 \frac{dy}{x+y}\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)}\mathbf{r}(y),$$
$$\mathbf{w}(x) = \mathbf{W}(it)\big|_{t=\frac{1}{\pi}\ln(1/x+\sqrt{1/x^2-1})}, \qquad t > 0.$$

Together with the characteristic parameter Λ, we introduce the spectral parameter $\mu = \Lambda^{-1}$ and the equation

$$(\mathbf{K}\mathbf{r})(x) = \mu\mathbf{r}(x) \tag{16}$$

in $L_2([0,1];\mathbb{C}^2)$.

It is now natural to study the properties of the operator $\mathbf{K}$ with a symmetric kernel that can be represented as

$$\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)} = \mathbf{a} + \mathbf{o}(1), \qquad (x,y) \to (0,0),$$

where, in applications, instead of $\mathbf{o}(1)$, a sharper estimate holds in the form $\mathbf{O}(x^b+y^b)$, where $(x,y) \to (0,0)$, $b > 0$, i.e., the matrix has elements decreasing like $\mathbf{O}(x^b + y^b)$ as $(x,y) \to (0,0)$. We also note that $\mathbf{w}(x)$ behaves like $O(1/\sqrt{1-x})$ as $x \to 1$. This shows that the following assertion is true.

Lemma 3.3. *The operator* $\mathbf{K} : L_2([0,1];\mathbb{C}^2) \to L_2([0,1];\mathbb{C}^2)$ *in* (16) *is bounded and self-adjoint.*

The operator $\mathbf{K}$ is called a perturbation of the bounded self-adjoint matrix operator $\mathbf{M}$, where $\mathbf{M}$ is the so-called Mehler operator[3] defined in $L_2([0,1];\mathbb{C}^2)$ by the expression

$$(\mathbf{M}\mathbf{r})(x) = \frac{1}{\pi}\int_0^1 \frac{dy}{x+y}\mathbf{a}\mathbf{r}(y).$$

Indeed, the operator $\mathbf{K}$ can be represented as

$$\mathbf{K} = \mathbf{M} + \mathbf{V} \tag{17}$$

according to the kernel representation

$$\frac{\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)}}{x+y} = \frac{\mathbf{a}}{x+y} + \frac{\mathbf{v}(x,y)}{x+y},$$

where $\mathbf{v}(x,y) := \sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)} - \mathbf{a}\mathbf{o}(1)$ (in our case, $\mathbf{O}(x^b+y^b)$) as $(x,y) \to (0,0)$, $b > 0$. The integral operator $\mathbf{V}$ in (17) is defined in $L_2([0,1];\mathbb{C}^2)$ as

$$(\mathbf{V}\mathbf{r})(x) = \frac{1}{\pi}\int_0^1 \frac{dy}{x+y}\mathbf{v}(x,y)\mathbf{r}(y).$$

In what follows, we assume that this operator $\mathbf{V}$ belongs to the Hilbert–Schmidt class S_2, which is ensured by the properties of the function $\mathbf{v}(x,y)$, in particular, if $\mathbf{v}(x,y) = \sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)} - \mathbf{a} = \mathbf{O}(x^b + y^b)$ as $x, y \to 0$.

[3]Here, we follow the terminology proposed by D. R. Yafaev.

In Appendix A, we study the spectral properties of the unperturbed Mehler operator $\mathbf{M}$ and explicitly diagonalize it. The content of the Appendix actually follows the well-known Mehler formulas obtained in 1881 [26]. The scalar operator M,

$$(Mu)(x) = \frac{1}{\pi}\int_0^1 \frac{dy}{x+y}u(y)$$

has a simple absolutely continuous spectrum $\sigma_a(M) = [0,1]$, and the eigenfunctions of the continuous spectrum have been obtained explicitly (see [27] and [28]):

$$\mathcal{P}_p(x) := \frac{\sqrt{p\tanh(\pi p)}}{x}P_{ip-1/2}(1/x),$$

where $P_{ip-1/2}(\,\cdot\,)$ is the Legendre function.

The operator $\mathbf{K}$ is a complex perturbation of the Mehler operator, which permits studying its spectral properties needed in investigating the FD equations by traditional methods.

4. Spectrum of the perturbed Mehler operator $\mathbf{K} = \mathbf{M} + \mathbf{V}$, discrete component of the spectrum of K, and its finiteness

We recall that we assumed the perturbation $\mathbf{V}$ to be of the Hilbert–Schmidt class S_2 and $\sigma_{\mathrm{e}}(\mathbf{M}) = [0,1]$. We use the Weyl theorem on the preservation of the essential spectrum under compact perturbations; as a result, we obtain the following assertion.

Lemma 4.1. *The essential spectrum $\sigma_{\mathrm{e}}(\mathbf{K})$ coincides with the interval $[0,1]$ if the perturbation $\mathbf{V}$ is compact.*

It follows from the properties of the kernel that the operator $\mathbf{K}$ is positive. Hence, it can have a discrete component of the spectrum only to the right of $\sigma_{\mathrm{e}}(\mathbf{K})$. The existence of a discrete component $\mathbf{K}$ is of special interest because each point μ_* of this component corresponds to the existence of a nontrivial solution from a given class for FD equation (11) for the corresponding $\Lambda_* = 1/\mu_*$. In this section, we consider some sufficient conditions formulated in terms of the perturbation $\mathbf{V}$ and, possibly, also with the help of the spectral characteristics of the operator $\mathbf{M}$. These sufficient conditions ensure that the discrete part of the spectrum $\sigma_{\mathrm{d}}(\mathbf{K})$ is not empty.

4.1. Simple sufficient condition for $\boldsymbol{\sigma_{\mathrm{d}}(\mathbf{K}) \neq \varnothing}$. We find $\mathbf{u}$ such that $(\mathbf{u},\mathbf{u})_{\mathcal{H}} = 1$ ($\mathcal{H} = L_2([0,1],\mathbb{C}^2)$) and

$$(\mathbf{Ku},\mathbf{u})_{\mathcal{H}} > 1. \tag{18}$$

Inequality (18) means that the discrete component of the spectrum is not empty. The strategy to verify the inequality is simple: we present a simple two-dimensional normalized vector for which inequality (18) holds under certain conditions on the potential. The choice of such a vector is directly related to a special form of the kernel of the integral operator $\mathbf{K}$. We take the normalized vector $\mathbf{u} = C(\mathbf{aw})^{-1/2}\mathbf{u^0}$, where $C = \|(\mathbf{aw})^{-1/2}\mathbf{u^0}\|^{-1}$ and $\mathbf{u^0} \in \mathbb{C}^2$ is a constant vector. We recall that $\mathbf{w} > 0$. Noting that

$$\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)} = (\sqrt{\mathbf{aw}(x)})^*\sqrt{\mathbf{aw}(y)},$$

we then find

$$(\mathbf{Ku},\mathbf{u}) = \frac{1}{\pi}\int_0^1 dx\int_0^1 dy\,\frac{\langle \mathbf{u}^0,\mathbf{u}^0\rangle_{\mathbb{C}^2}}{x+y} > 1,$$

where $\mathbf{u}^0 \in \mathbb{C}^2$ is a constant vector with complex coordinates $\mathbf{u}^0 = (r_1 + ig_1, r_2 + ig_2)^{\mathrm{T}}$, and hence the above inequality becomes

$$\frac{2\ln 2}{\pi}(r_1^2 + g_1^2 + r_2^2 + g_2^2) > 1. \tag{19}$$

The normalization condition takes the form

$$(\mathbf{u}, \mathbf{u})_{\mathcal{H}} = ((\mathbf{aw})^{-1}\mathbf{u}^0, \mathbf{u}^0)_{\mathcal{H}} = \int_0^1 dx\, \langle (\mathbf{aw})^{-1}(x)\mathbf{u}^0, \mathbf{u}^0 \rangle = \langle \mathbf{B}\mathbf{u}^0, \mathbf{u}^0 \rangle_{\mathbb{C}^2} = 1,$$

where the elements b_{ik} of a positive symmetric matrix $\mathbf{B} > 0$ are

$$b_{ik} := \int_0^1 dx\, \{(\mathbf{aw})^{-1}(x)\}_{ik}.$$

Finally, we write this condition in the form

$$b_{11}(r_1^2 + g_1^2) + 2b_{12}(r_1 r_2 + g_1 g_2) + b_{22}(r_2^2 + g_2^2) = 1. \tag{20}$$

It is natural to seek a vector $(r_1, g_1, r_2, g_2)^{\mathrm{T}} \in \mathbb{R}^4$ that provides the absolute maximum of the left-hand side of (19) under condition (20). This is the classical problem for a conditional extremum, which is solved by using Lagrange multipliers, one multiplier L in our case, and by introducing the Lagrange function

$$\mathcal{L}(r_1, g_1, r_2, g_2) = (r_1^2 + g_1^2 + r_2^2 + g_2^2) - L[b_{11}(r_1^2 + g_1^2) + 2b_{12}(r_1 r_2 + g_1 g_2) + b_{22}(r_2^2 + g_2^2) - 1].$$

Using the necessary condition for the extremum of $\mathcal{L}$ (i.e., $\nabla\mathcal{L} = 0$) that takes the form of a system of linear equations

$$L\mathbf{B}\mathbf{r} = \mathbf{r}, \qquad L\mathbf{B}\mathbf{g} = \mathbf{g},$$

we necessarily conclude that $\mathbf{r} = (r_1, r_2)^{\mathrm{T}}$ and $\mathbf{g} = (g_1, g_2)^{\mathrm{T}}$ must coincide with an eigenvector of the symmetric matrix $\mathbf{B} > 0$ in $\mathbb{R}^2$, with $L^{-1} =: \omega$ being a spectral parameter. An eigenvector corresponds to each of the two positive eigenvalues $\omega = \omega_{\min}(\mathbf{B})$ and $\omega = \omega_{\max}(\mathbf{B})$. We consider the minimal eigenvalue $\omega = \omega_{\min}$ and the corresponding eigenvector $\mathbf{e}$. To obtain the largest value in the left-hand side of (19), we take $\mathbf{r} = \mathbf{e}$ and $\mathbf{g} = \mathbf{e}$ and then substitute it in condition (20). From normalization condition (20), we then have $2\langle \mathbf{B}\mathbf{e}, \mathbf{e} \rangle = 1$ or

$$\omega_{\min}(\mathbf{B})(r_1^2 + g_1^2 + r_2^2 + g_2^2) = 1.$$

As a result, sufficient condition (19) becomes

$$\frac{2\ln 2}{\pi} > \omega_{\min}(\mathbf{B}). \tag{21}$$

Condition (21) is determined by the matrix $\mathbf{B}$, and is therefore in fact controlled by the potential $\mathbf{W}$. An analogue of this condition in the "scalar" case of one equation was used in [7], where it was verified numerically for a specific potential in the problem under study. In what follows, we apply condition (21) in the example of a meromorphic potential depending on the parameters of the considered problem and verify it numerically.

4.2. An alternative sufficient condition for $\sigma_{\mathbf{d}}(\mathbf{K}) \neq \varnothing$. As in Sec. 4.1, we start with condition (18) and substitute a normalized sequence $\mathbf{u}_n$, assuming that this condition is satisfied for some n and $\mathbf{u} = \mathbf{u}_n$. This would imply that the discrete component is not empty. To construct such a sequence, we consider a singular (Weyl) sequence $\mathbf{u}_n \in \mathcal{H}$, $n = 1, 2, \ldots$, corresponding to the point $\mu = 1$ of the essential spectrum of the Mehler operator $\mathbf{M}$. For example, $\|\mathbf{u}_n\| = 1$, $\mathbf{u}_n$ is an orthonormal sequence, i.e., $\mathbf{u}_n \rightharpoonup 0$ (weakly) such that $\|\mathbf{M}\mathbf{u}_n - \mathbf{u}_n\| \to 0$ as $n \to \infty$. Obviously, we obtain

$$([\mathbf{M}+\mathbf{V}]\mathbf{u}_n, \mathbf{u}_n) = \|\mathbf{u}_n\|^2 + ([\mathbf{M}+\mathbf{V}-\mathbf{I}]\mathbf{u}_n, \mathbf{u}_n) = 1 + ([\mathbf{M}+\mathbf{V}-\mathbf{I}]\mathbf{u}_n, \mathbf{u}_n),$$

and we conclude from (18) that the inequality

$$([\mathbf{M}+\mathbf{V}-\mathbf{I}]\mathbf{u}_n, \mathbf{u}_n) > 0, \tag{22}$$

which holds for some n, is a sufficient condition for the discrete spectrum to be nonempty for $\mu > 1$. Importantly, the partition of unity $\mathbf{E}_t$ for the Mehler operator $\mathbf{M}$ allows constructing a singular sequence explicitly.

We consider a numerical sequence $\varepsilon_n > 0$ ($\varepsilon_k > \varepsilon_{k+1}$), $\varepsilon_n \to 0$ (for example, $\varepsilon_n = 1/n$) and

$$\delta_n = (1-\varepsilon_n, 1-\varepsilon_{n+1}), \qquad |\delta_n| = \varepsilon_n - \varepsilon_{n+1}.$$

We choose an orthonormal sequence $\mathbf{u}_n$ such that $\mathbf{u}_n \in \mathbf{E}(\delta_n)\mathcal{H}$, noting that $\dim \mathbf{E}(\delta_n)\mathcal{H} = \infty$, where $\mathbf{E}(\delta) = \mathbf{E}_{b-0} - \mathbf{E}_{a+0}$, $\delta = (a, b)$. Using the spectral theorem, we then obtain

$$([\mathbf{M}-\mathbf{I}]\mathbf{u}_n, \mathbf{u}_n) = \int_{-\infty}^{\infty} (t-1)\, d(\mathbf{E}_t\mathbf{u}_n, \mathbf{u}_n) = \int_{\delta_n} (t-1)\, d(\mathbf{E}_t\mathbf{u}_n, \mathbf{u}_n),$$

because $\mathbf{E}_t\mathbf{u}_n = \mathbf{E}_t\mathbf{E}(\delta_n)\mathbf{h}/\|\mathbf{E}(\delta_n)\mathbf{h}\| = 0$ for $\mathbf{h} \in \mathcal{H}$ and $\delta_n \cap (-\infty, t) = \varnothing$. We also recall that $\sigma(\mathbf{M}) = [0, 1]$. We have

$$([\mathbf{V}+\mathbf{M}-\mathbf{I}]\mathbf{u}_n, \mathbf{u}_n) \geqslant \frac{1}{\pi}\int_0^1 dx\, \overline{\mathbf{u}}_n(x) \int_0^1 dy\, \frac{\mathbf{v}(x,y)}{y+x}\mathbf{u}_n(y) - \varepsilon_n(\mathbf{E}(\delta_n)\mathbf{u}_n, \mathbf{u}_n),$$

because $1 - t \leqslant \varepsilon_n$ for $t \in [1-\varepsilon_n, 1-\varepsilon_{n+1}]$ and

$$-\int_{\delta_n} (t-1)\, d(\mathbf{E}_t\mathbf{u}_n, \mathbf{u}_n) \leqslant \varepsilon_n(\mathbf{E}(\delta_n)\mathbf{u}_n, \mathbf{u}_n) = \varepsilon_n(\mathbf{u}_n, \mathbf{u}_n) = \varepsilon_n.$$

With (22) taken into account, we now obtain Theorem 2.1 in Sec. 2.2.

We introduce

$$\mathbf{Q}_n(x,y) = \mathbf{v}(x,y)\mathbf{u}_n(y) - \varepsilon_n \pi (x+y)\mathbf{u}_n(x).$$

Condition (6) can be equivalently written in the form

$$\frac{1}{\pi}\int_0^1 dx \int_0^1 dy\, \frac{\langle \mathbf{Q}_n(x,y), \mathbf{u}_n(x)\rangle_{\mathbb{C}^2}}{y+x} > 0 \tag{23}$$

for some $n = 1, 2, \ldots$, where $\mathbf{u}_n$ is the Weyl sequence corresponding to $\mu = 1$, with $\|\mathbf{u}_n\| = 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. Conditions (22), (21), or (19) are sufficient for the existence of a discrete component of the spectrum to the right of $[0, 1]$. We note that these conditions depend on the perturbation potential $\sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)} - \mathbf{a}$ and on the singular sequence $\mathbf{u}_n$, while it can be efficiently constructed because the spectral measure of the Mehler operator $\mathbf{M}$ is known. After some calculations, we obtain the sequence

$$\mathbf{u}_n = \mathbf{z}_n/\|\mathbf{z}_n\|,$$

which is orthonormal, and $\mathbf{z}_n = \mathbf{E}(\delta_n)\mathbf{h}$, $\mathbf{h} = (h^1, h^2)^{\mathrm{T}} \in \mathcal{H}$ and

$$\mathbf{z}_n(x) = (z_n^1(x), z_n^2(x))^{\mathrm{T}},$$

where

$$z_n^1(x) = \frac{1}{\pi}\int_{p_{n+1}}^{p_n} dp\,\mathcal{P}_p(x)(h^1)^*(p), \tag{24}$$

$$z_n^2(x) = \frac{1}{\pi}\int_{p_{n+1}/a}^{p_n/a} dp\,\mathcal{P}_p(x)(h^2)^*(p), \tag{25}$$

and

$$p_n = \frac{1}{\pi}\ln\biggl(\frac{1}{[1-\varepsilon_n]} + \sqrt{\frac{1}{[1-\varepsilon_n]^2} - 1}\biggr) = O(\sqrt{\varepsilon_n})$$

and $h^*(\,\cdot\,)$ is a modified Mehler–Fock transform of $h(\,\cdot\,)$ (see Appendix A).

We note that sufficient conditions are determined by the potential $\mathbf{w}$, which depends on parameters of the problem. In our example, these are $a = \gamma_2/\gamma_1$, Φ_1, and Φ_2. It is natural to expect (and this is verified numerically) that there exists a range of these parameters such that the discrete spectrum of $\mathbf{K}$ indeed exists. In a similar "scalar" problem [7], we also numerically verified a similar sufficient condition for some range of parameters.

4.3. Birman–Schwinger principle and the finiteness of the discrete spectrum. We now discuss the problem of the finiteness of the discrete spectrum. The approach used in this section is quite similar to that proposed for perturbations of the Carleman operator by Hankel operators in [25] and is based on the application of the Birman–Schwinger principle. In our case, this principle takes the form of the following theorem.

Theorem 4.1. *Let M_0 be a bounded self-adjoint operator such that $M_0 \leqslant 1$. Let $V_0 \geqslant 0$, and let $V_0 \in S_\infty$ (i.e., compact). Then the total number of eigenvalues (counted with multiplicities) of the operator $K_0 = M_0 + V_0$ that are greater than μ ($\mu \geq 1$) is equal to the total number of eigenvalues of the operator $B(\mu) = V_0^{1/2}[\mu - M_0]^{-1}V_0^{1/2}$.*

Using representation (50) for the resolvent, we obtain

$$\mathbf{B}(\mu) = \mu^{-1}(\mathbf{V} + \mathbf{V}^{1/2}\mathbf{A}_\mu\mathbf{V}^{1/2}). \tag{26}$$

We introduce the operator

$$\mathbf{Q} = Q\mathbf{I}, \qquad [Qf](t) = \langle \ln t\rangle f(t),$$

where $\langle \ln t\rangle = \ln(2/t)$, $f \in L_2(0,1)$. It is natural to define $\mathbf{Q}^\beta = Q^\beta\mathbf{I}$, $\beta \in \mathbb{R}$.

By (50), $\mathbf{A}_1$ is an integral operator with the kernel $\mathbf{a}(x, y; 1)$ admitting estimate (51) with $\mu = 1$. The following assertion holds.

Lemma 4.2. *Let $\alpha > 3/2$. Then*

$$\lim_{\mu\to 1}\|\mathbf{Q}^{-\alpha}(\mathbf{A}_\mu - \mathbf{A}_1)\mathbf{Q}^{-\alpha}\|_2 = 0 \tag{27}$$

in the Hilbert–Schmidt operator norm.

To prove this lemma, we have to verify that

$$\lim_{\mu\to 1}\int_0^1\int_0^1 \langle \ln t\rangle^{-2\alpha}\|\mathbf{a}(x,t,\mu)-\mathbf{a}(x,t,1)\|^2_{\mathbb{C}^2\to\mathbb{C}^2}\langle \ln x\rangle^{-2\alpha}\,dx\,dt = 0.$$

By Lebesgue's dominated convergence theorem, it suffices to prove that there exists an integrable majorant of the integrand. By estimate (51), the majorant has the form

$$C\langle \ln t\rangle^{-2\alpha}\frac{\langle \ln t\rangle^2\langle \log x\rangle^2}{tx}\langle \ln x\rangle^{-2\alpha}$$

and obviously belongs to $L_1((0,1)\times(0,1))$ for $2\alpha > 3$.

Lemma 4.2 implies the following assertion.

Lemma 4.3. *Let $\alpha > 3/2$ and let a self-adjoint operator $\mathbf{V} \geq 0$, $\mathbf{V}\in S_2$, be of Hilbert–Schmidt class; in addition, let the operator $\mathbf{Q}^\alpha\mathbf{V}\mathbf{Q}^\alpha \in S_\infty$ be compact. Then the operator $\mathbf{B}(\mu)$ in (26) has the limit*

$$\mathbf{B}(1) = \mathbf{V} + \mathbf{V}^{1/2}\mathbf{A}_1\mathbf{V}^{1/2} \tag{28}$$

in the Hilbert–Schmidt norm $\|\cdot\|_2$ as $\mu\to 1$.

The proof follows from the chain of relations

$$\begin{aligned}\|\mathbf{B}(\mu)-\mathbf{B}(1)\|_2^2 &= \|\mathbf{V}^{1/2}\mathbf{Q}^\alpha\mathbf{Q}^{-\alpha}(\mathbf{A}_\mu-\mathbf{A}_1)\mathbf{Q}^{-\alpha}\mathbf{Q}^\alpha\mathbf{V}^{1/2}\|_2^2 =\\ &= \|\mathbf{Q}^\alpha\mathbf{V}\mathbf{Q}^\alpha\mathbf{Q}^{-\alpha}(\mathbf{A}_\mu-\mathbf{A}_1)\mathbf{Q}^{-\alpha}\|_2^2,\end{aligned}$$

where $\|A\|_2^2 = \langle A, A\rangle_{S_2}$, the relation $\langle A,B\rangle_{S_2} := \mathrm{Tr}(B^*A) = \mathrm{Tr}(AB^*)$, and Lemma 4.2.

Let $N(\mu)$ be the total number of eigenvalues of the operator $\mathbf{K} = \mathbf{M}+\mathbf{V}$ located to the right of μ, $\mu \geqslant 1$. It follows from the Birman–Schwinger principle that $N(\mu) \leq \|\mathbf{B}(\mu)\|_2^2$, and then Lemma 4.3 implies the estimate

$$N(1) \leqslant \|\mathbf{B}(1)\|_2^2.$$

Using (28), we further obtain

$$\|\mathbf{B}(1)\|_2 \leqslant (\|\mathbf{V}\|_2 + \|\mathbf{Q}^\alpha\mathbf{V}\mathbf{Q}^\alpha\|\|\mathbf{Q}^{-\alpha}\mathbf{A}_1\mathbf{Q}^{-\alpha}\|).$$

Using the properties of the kernel $\mathbf{a}(x,t,1)$, we introduce

$$G_\alpha^2 := \int_0^1\int_0^1 \langle \ln t\rangle^{-2\alpha}\|\mathbf{a}(x,t,1)\|^2_{\mathbb{C}^2\to\mathbb{C}^2}\langle \ln x\rangle^{-2\alpha}\,dx\,dt.$$

This leads to the estimate

$$N(1) \leqslant (\|\mathbf{V}\|_2 + G_\alpha\|\mathbf{Q}^\alpha\mathbf{V}\mathbf{Q}^\alpha\|)^2 \tag{29}$$

and to Theorem 2.2 in Sec. 2.2.

We note that if we replace the condition $\alpha > 3/2$ in Theorem 2.2 with $\alpha \leqslant 3/2$, then the spectrum to the right of $\mu = 1$ is infinite.

5. Spectral properties of FD equation (11)

We apply the results obtained above for the perturbed Mehler operator $\mathbf{M}+\mathbf{V}$ to the system of FD equations. We now assume that one of sufficient conditions (23) or (21) (also see (6)) is satisfied and $\mu_m = \Lambda_m^{-1} \in \sigma_{\rm d}(\mathbf{M}+\mathbf{V})$ is the eigenvalue corresponding to an eigenfunction $\mathbf{r}_m(x)$. Then the corresponding solution $\mathbf{h}_m(\nu) = \mathbf{W}^{-1/2}(\nu)\,\mathbf{r}_m(x)|_{x=1/\cos(\pi\nu)}$ (see (13)) satisfies Eq. (12) for $\Lambda = \Lambda_m$. Moreover, as was described above, the solution $\mathbf{h}_m$ can be extended as a meromorphic function, is a solution of Eq. (11), and admits the estimate

$$\int_0^{i\infty} |\sin\pi\tau|\,\|\sqrt{\mathbf{W}(\tau)}\mathbf{h}_m(\tau)\|^2_{\mathbb{C}^2}\,|d\tau| < \infty. \tag{30}$$

Here, we also show that together with estimate (30), the solution $\mathbf{h}_m$ satisfies the following estimate on the imaginary axis:

$$\mathbf{h}_m(\nu) = O(e^{i\nu[\pi-\tau_m]}), \qquad \nu \to i\infty, \tag{31}$$

where $\tau_m \in (0,\pi/2)$ can is uniquely defined by the equation $\sin\tau_m = \Lambda_m$. Asymptotic estimate (31) is used in what follows to obtain the asymptotics of an eigenfunction of $A_{\rm s}$.

We use the Fourier transformation along the imaginary axis

$$\chi(\zeta) = \int_{i\mathbb{R}} e^{i\zeta\nu} h(\nu)\,d\nu, \qquad h(\nu) = -\frac{\text{v.p.}}{2\pi}\int_{i\mathbb{R}} e^{-i\zeta\nu}\chi(\zeta)\,d\zeta$$

and take Eq. (11) into account. For the Fourier transform $\mathbf{H}(\,\cdot\,)$ of $\mathbf{h}(\,\cdot\,)$, we obtain

$$[\sin\zeta\mathbf{I} - \Lambda\mathbf{a}]\mathbf{H}(\zeta) + \Lambda\int_{i\mathbb{R}} e^{i\zeta\nu}[\mathbf{a}\mathbf{W}(\nu) + \mathbf{a}]\mathbf{h}(\nu)\,d\nu = 0,$$

and then recalling that $\Lambda = \Lambda_m \in (0,1)$, we introduce $\tau_m \in (0,\pi/2)$, $\sin\tau_m := \Lambda_m$ and obtain the relation

$$\mathbf{H}(\zeta) = -\sin\tau_m[\sin\zeta\mathbf{I} - \sin\tau_m\mathbf{a}]^{-1}\int_{i\mathbb{R}} e^{i\zeta\nu}[\mathbf{a}\mathbf{W}(\nu) + \mathbf{a}]\mathbf{h}(\nu)\,d\nu. \tag{32}$$

Representation (32) allows defining a strip on the complex ζ plane where $\mathbf{H}(\zeta)$ is holomorphic. We know that $\mathbf{h}(\nu)$ admits the estimate $|\mathbf{h}(\nu)| \leqslant Ce^{-\pi|\nu|/2}$ on the imaginary axis as $\nu \to \pm i\infty$, which means that the function $\mathbf{H}(\zeta)$ is holomorphic in the strip $\Pi(-\pi/2,\pi/2) := \{\zeta \in \mathbb{C} : -\pi/2 < \mathrm{Re}(\zeta) < \pi/2\}$. Our goal is to show that it is holomorphic in a wider strip, namely, in $\Pi(-\pi+\tau_m, \pi-\tau_m)$, where $\tau_m \in (0,\pi/2)$ is defined above. We note that $\mathbf{H}(\zeta)$ is even because $\mathbf{h}(\nu)$ is even. As a result, it suffices to define the regularity strip of the integral in the right-hand side of (32) only for $\mathrm{Re}(\zeta) > 0$. We consider the right-hand side of representation (32). Obviously, the matrix $[\sin\zeta\mathbf{I} - \sin\tau_m\mathbf{a}]^{-1}$ has simple poles at $\zeta = \pi - \tau_m$ in the first row, and at $\zeta = \pi - t_m$ in the second row, where $\sin t_m = a\sin\tau_m$ and $t_m \in (0,\pi/2)$, $t_m \leqslant \tau_m$. We note that these poles are the nearest to the imaginary axis for this factor in the right-hand side of (32). We now consider the integral in the right-hand side of (32). It turns out to be holomorphic in a strip wider than $\Pi(-[\pi-\tau_m], \pi-\tau_m)$. To verify this, we use the representation ($\mathrm{Re}(\zeta) > 0$) for

$$\begin{aligned}\int_{i\mathbb{R}} e^{i\zeta\nu}[\mathbf{a}\mathbf{W}(\nu) + \mathbf{a}]\mathbf{h}(\nu)\,d\nu = &\int_{i\mathbb{R}} e^{i\zeta\nu}[\mathbf{a}\mathbf{W}(\nu) + i\mathbf{a}\tan(b\nu)]\mathbf{h}(\nu)\,d\nu + \\ &+ \int_{i\mathbb{R}} e^{i\zeta\nu}[-i\mathbf{a}\tan(b\nu) + \mathbf{a}\,\mathrm{sign}(i\nu)]\mathbf{h}(\nu)\,d\nu + \\ &+ \int_{i\mathbb{R}} e^{i\zeta\nu}[-\mathbf{a}\,\mathrm{sign}(i\nu) + \mathbf{a}]\mathbf{h}(\nu)\,d\nu.\end{aligned}$$

In the last term, the integral is taken along $i\mathbb{R}_+$, because $[-\mathbf{a}\,\mathrm{sign}(i\nu)+\mathbf{a}] = 0$ on the negative part of the axis and hence the integral is holomorphic for $\mathrm{Re}(\zeta) > 0$. Assuming that the asymptotics $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{O}(e^{\pm iq_*\nu})$ ($q_* = 2\min\{\Phi_2 - \Phi_1, \Phi_1, \pi - \Phi_2\} > 0$ in our example below) as $\nu \to \pm i\infty$ holds on the imaginary axis, we see that the first integral is regular in the strip $\Pi(-\pi + \tau_m - q_*, \pi - \tau_m + q_*)$; we assume that $b > q_*$. If we take $b > 0$ sufficiently large, we can conclude that the second integral is holomorphic in $\Pi(-[2b + \pi - \tau_m], 2b + \pi - \tau_m)$. We see that $\mathbf{H}(\,\cdot\,)$ is a holomorphic function in $\Pi(-\pi + \tau_m, \pi - \tau_m)$; using the inverse Fourier transformation, we then obtain the following assertion.

Lemma 5.1. *The vector* $\mathbf{H}(\,\cdot\,)$ *is holomorphic in the strip* $\Pi(-\pi + \tau_m, \pi - \tau_m)$ *and* $\mathbf{h}(\nu)$ *then has the asymptotics*

$$\mathbf{h}(\nu) = O(e^{i\nu[\pi - \tau_m]}), \qquad \nu \to i\infty,$$

in a neighborhood of the imaginary axis if $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{O}(e^{iq_*\nu})$, $q_* > 0$.

We note that Lemma 5.1 is also true if the constraints on the behavior of the potential at infinity are weaker, for example, if $\mathbf{W}(\nu) = \mathbf{I} + \mathbf{O}(1/\nu)$. The estimate $\|\mathbf{H}(\zeta)\| < Ce^{-\alpha_0|\zeta|}$, $\alpha_0 > 1$ and $\mathrm{Im}(\zeta) \to \infty$, is also true because the function $\mathbf{h}$ is holomorphic in the strip $\Pi(-1 - \delta, 1 + \delta)$.

We now describe the characteristic set $C_\mathrm{d} \cup C_\mathrm{e}$ of the values of Λ for Eq. (11), i.e., those values of Λ for which the equation has a nontrivial solution from the corresponding class. The set of characteristics values C_d is nonempty and finite if and only if this is true for $\sigma_\mathrm{d}(\mathbf{K})$. The statements from the preceding sections describe conditions sufficient for this.[4] By definition, $\Lambda_m = 1/\mu_m$ belongs to the set C_d of characteristic values of Eq. (11) if $\mu_m = (\sin\tau_m)^{-1} \in \sigma_\mathrm{d}(\mathbf{K})$. Obviously, $C_\mathrm{d} \subset [0, 1]$. Similarly, $\Lambda \in C_\mathrm{e} = [1, \infty)$, i.e., by definition, it belongs to the essential characteristic set if $\mu = \Lambda^{-1} \in \sigma_\mathrm{e}(\mathbf{K}) = [0, 1]$, where $\mathbf{M} + \mathbf{V}$ is a perturbation of the Mehler operator related to Eq. (11) with the potential $\mathbf{W} \in \mathcal{W}$. In this case, we have $|H(\nu)| < \mathrm{const}\,|e^{i\nu\pi/2)|}$, $\nu \to i\infty$, $\nu \in \Pi_{1+\delta}$. Using the results in the preceding sections, we obtain the following assertion.

Proposition 5.1. *The set* C_d *formed by characteristic values of Eq.* (11) *is nonempty if the potential* $\mathbf{w}(x) = \mathbf{W}|_{x=1/\cos\pi\nu}$ *(and the potential* $\mathbf{v}(x, y) := \sqrt{\mathbf{w}(x)}\mathbf{a}\sqrt{\mathbf{w}(y)} - \mathbf{a}$*) satisfies sufficient conditions* (21) *or* (23), (6). *Under the conditions of Theorem* 2.2, *this set is finite. Estimates* (31) *hold for the corresponding solutions, and the solutions are of class* $\mathcal{M}$.

5.1. Solvability of an FD equation for characteristic values and eigenvalues of the operator $\boldsymbol{A}_\mathrm{s}$. We now use the sufficient condition obtained above for the existence of characteristic numbers of Eq. (11) and verify this condition for a domain of parameters. Condition (21) is determined by the matrix $\mathbf{B}$ and hence in fact by the potential $\mathbf{W}$. An analog of this condition in the "scalar" case is used in [7], where it was verified numerically for some domain of parameters of the problem.

Table 1. Values $F(\Phi_1) = 2\ln 2/\pi - \omega_{\min}(\mathbf{B}[\Phi_1])$. The discrete spectrum exists for $F(\Phi_1) > 0$ in accordance with condition (21). For example, $a := \gamma_2/\gamma_1 = 0.5$, $\Phi_2 = 2\pi/3 \approx 2.094$ (see Fig. 1)

	1	2	3	4	5	6	7	8	9
Φ_1	1.885	1.676	1.466	1.257	1.047	0.8378	0.5236	0.3142	0.1047
$F(\Phi_1)$	0.0506	0.0299	0.0125	−0.0008	−0.0092	−0.012	−0.0043	0.0092	0.029

A similar verification of sufficient condition (21) for the potential $\mathbf{W}(\nu)$ is presented in Table 1, which shows the results of calculations of $F(\Phi_1) = 2\ln 2/\pi - \omega_{\min}(\mathbf{B}[\Phi_1])$ depending on Φ_1; positive values then correspond to the case where the discrete component $\sigma_\mathrm{d}(\mathbf{K})$ is nonempty. The parameter $\Phi_2 = 2\pi/3$ is

[4] Solutions (11) corresponding to $\Lambda = \mu^{-1}$ and $\mu \in \sigma_\mathrm{e}(M + V)$, i.e., to the essential spectrum, can also be described.

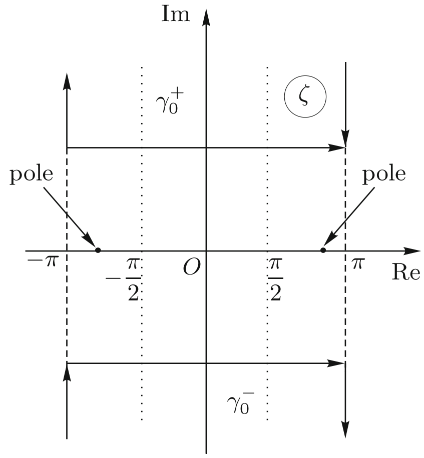

Fig. 2. Integration contour $\gamma_0 = \gamma_0^+ \cup \gamma_0^-$ and singularities.

fixed and $a = 0.5$. Obviously, we have the range of Φ_1 values for which condition (21) is not satisfied, i.e., $F(\Phi_1) < 0$. Thus, there exists a set of Λ_m values, $m = 1, 2, \ldots, N_l$ (which is finite in our case by Theorem 2.2 applied to the potential $\mathbf{W}(\nu)$), and the eigenvalues E_m of the operator A_s are given by

$$E_m = -\left[\frac{\gamma_1}{2\Lambda_m}\right]^2.$$

We now proceed to studying the asymptotics of the eigenfunction of A_s corresponding to the eigenvalue E_m.

6. Asymptotics of the eigenfunction

6.1. Sommerfeld integral representations for the eigenfunctions of A_s. To verify that representation (8) is an eigenfunction, we study its behavior as $r \to \infty$. Calculating the asymptotics, we show that it decreases exponentially. But the direct replacement of the MacDonald function with its asymptotics in the integrand $K_\nu(\kappa r) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-\kappa r}}{\kappa r}$ makes the integral divergent, and hence this direct way does not lead to the desired result. Instead, we use the Sommerfeld integral representation

$$K_\nu(\kappa r) = \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa r \cos\zeta} \frac{\sin \nu\zeta}{\sin \pi\nu}$$

with the integration contour shown in Fig. 2, substitute it in the integral, and then change the order of integrations. In Ω_1, Ω_2, and Ω_3, we obtain

$$\begin{aligned} u_{1m}(r,\varphi) &= \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa_m r \cos\zeta} F_1(\zeta,\varphi), \\ u_{2m}(r,\varphi) &= \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa_m r \cos\zeta} F_2(\zeta,\varphi), \\ u_{3m}(r,\varphi) &= \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa_m r \cos\zeta} F_3(\zeta,\varphi). \end{aligned} \tag{33}$$

The angles have the respective values Φ_1, $\Phi_2-\Phi_1$, and $\pi-\Phi_2$, and

$$\begin{aligned}
F_1(\zeta,\varphi) &= \frac{1}{2i}\int_{-i\infty}^{i\infty}\sin(\zeta\nu)\frac{\cos(\nu\varphi)}{\cos(\Phi_1\nu)}H_{1m}(\nu)\,d\nu,\\
F_2(\zeta,\varphi) &= \frac{1}{2i}\int_{-i\infty}^{i\infty}\sin(\zeta\nu)\biggl(\frac{\cos(\nu[\Phi_2-\varphi])}{\cos(\nu[\Phi_2-\Phi_1])}H_m(\nu)\,d\nu+\\
&\qquad\qquad+\frac{\sin(\nu[\Phi_1-\varphi])}{\sin(\nu[\Phi_1-\Phi_2])}\tilde{h}_m(\nu)\biggr)\,d\nu,\\
F_3(\zeta,\varphi) &= \frac{1}{2i}\int_{-i\infty}^{i\infty}\sin(\zeta\nu)\frac{\cos(\nu[\pi-\varphi])}{\cos(\nu[\pi-\Phi_2])}H_{3m}(\nu)\,d\nu.
\end{aligned}\tag{34}$$

We omit the index m in the notation unless this leads to a confusion.

We study the asymptotic behavior of Sommerfeld integrals (33) under the assumption that the integrands in (34) are known after solving FD equations (11). The idea of such an investigation is traditional. In the framework of the saddle-point method, the integration contour γ_0 in (33) must be deformed into the steepest descent contour $\gamma_0^{\pm\pi}=\{\zeta\colon \mathrm{Re}(\zeta)=\pm\pi\}$. In the process of such a deformation, some singularities of the integrand (poles) can be captured, which then contribute to the asymptotics along with the saddle points $\zeta=\pm\pi$. It is therefore necessary to investigate the behavior of the functions F_1, F_2, and F_3 on the complex ζ plane and, in particular, to determine the singularities that are captured in the process contour deformation.

6.2. Analytic properties of $F_1(\zeta,\varphi)$, $F_2(\zeta,\varphi)$, and $F_3(\zeta,\varphi)$. It is convenient to transform integral representations (34) starting with $F_1(\zeta,\varphi)$,

$$\begin{aligned}
F_1(\zeta,\varphi) &= \frac{1}{2i}\int_{-i\infty}^{i\infty}\frac{1}{2}\biggl(\frac{\sin(\nu[\zeta+\varphi])}{\cos(\Phi_1\nu)}+\frac{\sin(\nu[\zeta-\varphi])}{\cos(\Phi_1\nu)}\biggr)H_1(\nu)\,d\nu=\\
&= f_1(\zeta+\varphi)+f_1(\zeta-\varphi),
\end{aligned}\tag{35}$$

where

$$f_1(\zeta)=\frac{1}{4i}\int_{-i\infty}^{i\infty}\frac{\sin(\nu\zeta)}{\cos(\Phi_1\nu)}H_1(\nu)\,d\nu=\frac{1}{4i}\int_{-i\infty}^{i\infty}\frac{e^{i\nu\zeta}}{i\cos(\Phi_1\nu)}H_1(\nu)\,d\nu\tag{36}$$

and the function $f_1(z)$ is odd, $f_1(z)=-f_1(-z)$. In the same way, we obtain

$$\begin{aligned}
F_3(\zeta,\varphi) &= \frac{1}{2i}\int_{-i\infty}^{i\infty}\frac{1}{2}\biggl(\frac{\sin(\nu[\zeta+(\pi-\varphi)])}{\cos(\nu[\pi-\Phi_2])}+\frac{\sin(\nu[\zeta-(\pi-\varphi)])}{\cos(\nu[\pi-\Phi_2])}\biggr)H_3(\nu)\,d\nu=\\
&= f_3(\zeta+[\pi-\varphi])+f_3(\zeta-[\pi-\varphi]),
\end{aligned}\tag{37}$$

where

$$f_3(\zeta)=\frac{1}{4i}\int_{-i\infty}^{i\infty}\frac{\sin(\nu\zeta)}{\cos(\nu[\pi-\Phi_2])}H_3(\nu)\,d\nu=\frac{1}{4i}\int_{-i\infty}^{i\infty}\frac{e^{i\nu\zeta}}{i\cos(\nu[\pi-\Phi_2])}H_3(\nu)\,d\nu\tag{38}$$

and the function $f_3(z)$ is odd, $f_3(z)=-f_3(-z)$. We also have

$$\begin{aligned}
F_2(\zeta,\varphi) &= \frac{1}{2i}\int_{-i\infty}^{i\infty}d\nu\,\sin(\nu\zeta)(\cos(\nu\varphi)Q_1(\nu)+\sin(\nu\varphi)iQ_2(\nu))=\\
&= \frac{1}{2i}\int_{-i\infty}^{i\infty}d\nu\,\sin(\nu\zeta)e^{i\nu\varphi}(Q_1(\nu)+Q_2(\nu)),
\end{aligned}\tag{39}$$

where

$$Q_1(\nu) = \frac{\cos(\nu\Phi_2)}{\cos(\nu[\Phi_2 - \Phi_1])} H(\nu) + \frac{\sin(\nu\Phi_1)}{\sin(\nu[\Phi_1 - \Phi_2])} \tilde{h}(\nu),$$
$$Q_2(\nu) = \frac{1}{i}\left(\frac{\sin(\nu\Phi_2)}{\cos(\nu[\Phi_2 - \Phi_1])} H(\nu) - \frac{\cos(\nu\Phi_1)}{\sin(\nu[\Phi_1 - \Phi_2])} \tilde{h}(\nu) \right).$$

It follows from (39) that

$$F_2(\zeta, \varphi) = f_2(\zeta + \varphi) - f_2(-\zeta + \varphi),$$

where

$$f_2(\zeta) = \frac{1}{4i} \int_{-i\infty}^{i\infty} e^{i\nu\zeta} \frac{1}{i} (Q_1(\nu) + Q_2(\nu))\, d\nu. \tag{40}$$

Obviously, it follows from representations (35)–(40) that the functions f_1, f_2, and f_3 are the Fourier transforms along the imaginary axis of some meromorphic functions on the complex ν plane and are meromorphic. As a result, the asymptotics of H_1, H, H_3, and $\tilde{h}$ as $\nu \to i\infty$ are related to the location of singularities (poles) of the functions f_1, f_2, and f_3. As was noted above, the poles of f_1, f_2, and f_3 can contribute to the asymptotics. From (33), we have

$$\begin{aligned} u_1(r, \varphi) &= \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa r \cos\zeta} (f_1(\zeta + \varphi) - f_1(-\zeta + \varphi)), \qquad \varphi \in [0, \Phi_1], \\ u_2(r, \varphi) &= \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa r \cos\zeta} (f_2(\zeta + \varphi) - f_2(-\zeta + \varphi)), \qquad \varphi \in [\Phi_1, \Phi_2], \\ u_3(r, \varphi) &= \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa r \cos\zeta} (f_3(\zeta + [\pi - \varphi]) - f_3(-\zeta + [\pi - \varphi])), \qquad \varphi \in [\Phi_2, \pi]. \end{aligned} \tag{41}$$

To describe the singularities of the functions f_1, f_2, and f_3 in (36), (38), and (40), we first find the corresponding strips on the complex plane, where these meromorphic functions are holomorphic. We first calculate the higher-order terms of the asymptotics of H_1, H, H_3, and $\tilde{h}$ as $\nu \to i\infty$. This can be done using the asymptotics of h_1 and h_2, which are solutions of Eq. (11). We recall that these solutions directly determine H_1, H, H_3, $\tilde{h}$ by explicit linear relations (9). From these formulas we obtain the $\nu \to i\infty$ asymptotics in the strip $\Pi(-\delta, \delta)$ for some $\delta > 0$:

$$\begin{aligned} H_m(\nu) &= O(e^{i\nu[\pi - \tau_m]}), \\ H_{1m}(\nu) &= O(e^{i\nu[\pi - \tau_m]}), \\ H_{3m}(\nu) &= O(e^{i\nu[\pi - \tau_m]}), \\ \tilde{h}_m(\nu) &= O(e^{i\nu[\pi - \tau_m]}). \end{aligned}$$

Because the meromorphic functions f_1, f_2, and f_3 are related to H_1, H, H_3, and $\tilde{h}$ by Fourier-type transformations (36), (38), and (40), it follows from Lemma 5.1 that the following assertion holds.

Lemma 6.1. *The meromorphic functions f_1, f_2, and f_3 are holomorphic in the respective strips* $\Pi(-[\pi - \tau_m + \Phi_1], \pi - \tau_m + \Phi_1)$, $\Pi(-[\pi - \tau_m + \Phi_1], \pi - \tau_m + \Phi_2)$, *and* $\Pi(-[\pi - \tau_m + \pi - \Phi_2], \pi - \tau_m + \pi - \Phi_2)$ *on the complex ζ plane.*

We note that f_2 has the following representation in the strip $\Pi(-[\pi - \tau_m] + \Phi_1, [\pi - \tau_m] + \Phi_2)$:

$$f_2(\zeta) = \frac{1}{4i} \int_{-i\infty}^{i\infty} \frac{1}{i} \left(\frac{H(\nu) e^{i\nu[\zeta - \Phi_2]}}{\cos(\nu[\Phi_2 - \Phi_1])} + \frac{i\tilde{h}(\nu) e^{i\nu[\zeta - \Phi_1]}}{\sin(\nu[\Phi_1 - \Phi_2])} \right) d\nu.$$

We conclude from Lemma 6.1 that $f_1(\zeta)$ has poles at the points $\zeta = \pm[\pi - \tau_m + \Phi_1]$ and $f_3(\zeta)$ has poles at $\zeta = \pm[\pi - \tau_m + \pi - \Phi_2]$, which are the singularities closest to the imaginary axis, while $f_2(\zeta)$ has the same poles at the points $\zeta = -[\pi - \tau_m] + \Phi_1$ and $\zeta = [\pi - \tau_m] + \Phi_2$. We recall that in the strips located between these poles and parallel to the imaginary axis, the functions f_1, f_2, and f_3 are regular (holomorphic). The other poles are also real.

The further constructions and derivations of Maliuzhinets functional equations in our problem are completely similar to those proposed in [7]. In particular, the Maliuzhinets equations given in Appendix B allow continuing the transforms of f_1, f_2, and f_3 from the holomorphy strips into the complex plane and verifying that their poles are real.

6.3. Poles of the Sommerfeld transforms of f_1, f_2, and f_3 and the asymptotics of integrals. To calculate the asymptotics of Sommerfeld integrals in (41), according to the well-known procedure, we deform the integration contour γ_0 in Fig. 2 into the steepest descent contour $\gamma_0^{\pm\pi} = \{\zeta : \mathrm{Re}(\zeta) = \pm\pi\}$ that passes through the respective point $\pm\pi$. In the process of deformation, the poles of the integrands in (41) are captured. The location of the poles depends on the observation angle φ, but they are located outside the closed strip $\Pi(-\pi/2, \pi/2)$. This means that the captured poles generate exponentially small terms in the asymptotics as $r \to \infty$. This is so because $e^{\kappa r \cos\zeta}$ in the integrand in (41) decreases if $\pi/2 < |\mathrm{Re}(\zeta)| < 3\pi/2$ and all the poles captured inside this strip generate decreasing exponentials. An additional fast decreasing contribution (of the order $O(e^{-\kappa r}/\sqrt{r})$) is made by the saddle points at $\zeta = \pm\pi$ (see [29]). But for some observation angles φ, some poles $\zeta_p(\varphi)$ of the transforms can be located in a narrow neighborhood of saddle points and cross them as φ varies. This means that the asymptotic estimate of the integral must be modified. In this case, the asymptotics is expressed in terms of a Fresnel-type integral. The directions φ for which such a collision of a saddle point and a pole occurs are said to be *singular*. In these directions, the asymptotically decreasing regime is switched over from one to another. We now consider the corresponding calculations in more detail.

We consider u_1 in (41) ($0 \leqslant \varphi \leqslant \Phi_1$) and rewrite it as

$$u_1(r, \varphi) = \frac{1}{2i\pi} \int_{\gamma_0} d\zeta\, e^{\kappa r \cos\zeta} 2 f_1(\zeta + \varphi).$$

Near the pole at $-\varphi + [\pi - \tau_m + \Phi_1]$, the transform f_1 becomes

$$f_1(\zeta + \varphi) = \frac{A_1^+}{\zeta + \varphi - [\pi - \tau_m + \Phi_1]} + \cdots .$$

This pole intersects the saddle point π and is captured if φ ranges the interval $[0, \Phi_1]$ and $\tau_m < \Phi_1$. This means that, by the residue theorem, a contribution appears if

$$\frac{\pi}{2} < -\varphi + [\pi - \tau_m + \Phi_1] \leqslant \pi - \frac{C}{(\kappa r)^{1/2+\epsilon}}, \qquad C > 0,\ \kappa r \to \infty.$$

We have ($\varphi + \tau_m - \Phi_1 > C/(\kappa r)^{1/2+\epsilon}$)

$$u_1(r, \varphi) = 2A_1^+ e^{-\kappa r \cos(\varphi + \tau_m - \Phi_1)} + u_1^*(r, \varphi) + \cdots , \tag{42}$$

where the contribution due to the saddle points $\pm\pi$ has the form

$$u_1^*(r, \varphi) = 2[f_1(-\pi + \varphi) - f_1(\pi + \varphi)] \frac{e^{-\kappa r}}{\sqrt{2\pi\kappa r}} \left(1 + O\left(\frac{1}{\kappa r}\right)\right).$$

We note that this pole is not close to π, i.e., φ is outside the domain

$$|\varphi + \tau_m - \Phi_1| \leqslant O\biggl(\frac{1}{(\kappa r)^{1/2+\epsilon}}\biggr), \tag{43}$$

which describes a neighborhood of the singular direction $\varphi = \Phi_1 - \tau_m$, for a small $\epsilon > 0$ and for $\tau_m \in (0, \Phi_1)$. But if $\pi/2 > \tau_m > \Phi_1$, then the corresponding pole does not cross the saddle point π and there is no singular direction related to this pole. The dots in (42) mean that there can be a contribution from other poles that can be captured in the process deforming γ_0 into the saddle point contours $\gamma_0^{\pm\pi}$. Obviously, the set of captured poles also depends on Φ_1, Φ_2. The transform f_1 can thus be represented as

$$f_1(\zeta + \varphi) = \frac{A_1^-}{\zeta + \varphi + [\pi - \tau_m + \Phi_1]} + \cdots$$

in a neighborhood of the pole $\zeta = -\varphi - [\pi - \tau_m + \Phi_1]$. This pole contributes to the asymptotics if

$$-\frac{\pi}{2} > -\varphi - [\pi - \tau_m + \Phi_1] \geqslant -\pi + \frac{C}{(\kappa r)^{1/2+\epsilon}}, \qquad C > 0,\ \kappa r \to \infty,$$

or $\varphi - \tau_m + \Phi_1 \leqslant -C/(\kappa r)^{1/2+\epsilon}$. This inequality holds for $\tau_m > \Phi_1$ and for some $\varphi \in (0, \Phi_1)$. We have

$$\begin{aligned} u_1(r,\varphi) = {}& 2H(\varphi + \tau_m - \Phi_1)A_1^+ e^{-\kappa r\cos(\varphi + \tau_m - \Phi_1)} + \\ & + 2[f_1(-\pi + \varphi) - f_1(\pi + \varphi)]\frac{e^{-\kappa r}}{\sqrt{2\pi\kappa r}}\biggl(1 + O\biggl(\frac{1}{\kappa r}\biggr)\biggr) + \cdots \end{aligned} \tag{44}$$

for $\tau_m < \Phi_1$ and $|\varphi + \tau_m - \Phi_1| \geqslant O(1/(\kappa r)^{1/2+\epsilon})$.

For $\tau_m > \Phi_1$, we obtain

$$\begin{aligned} u_1(r,\varphi) = {}& 2H(-[\varphi - \tau_m + \Phi_1])A_1^- e^{-\kappa r\cos(\varphi - \tau_m + \Phi_1)} + \\ & + 2[f_1(-\pi + \varphi) - f_1(\pi + \varphi)]\frac{e^{-\kappa r}}{\sqrt{2\pi\kappa r}}\biggl(1 + O\biggl(\frac{1}{\kappa r}\biggr)\biggr) + \cdots \end{aligned} \tag{45}$$

for $|\varphi - \tau_m + \Phi_1| \geqslant O(1/(\kappa r)^{1/2+\epsilon})$, $H(\cdot)$ is the Heaviside function. The dots in asymptotics (44) and (45) corresponds to the contribution of other possibly captured poles, which decreases faster than the terms calculated explicitly. The asymptotics in (44) and (45) are not uniform in φ. If the inequalities $|\varphi + \tau_m - \Phi_1| \geqslant O(1/(\kappa r)^{1/2+\epsilon})$ for (44) or $|\varphi - \tau_m + \Phi_1| \geqslant O(1/(\kappa r)^{1/2+\epsilon})$ for (45) are violated, then the asymptotic expressions must be modified by using Fresnel-type integrals, as was mentioned above.

6.4. Asymptotics near singular directions. For definiteness, we assume that $\tau_m < \Phi_1$ and consider a neighborhood of singular directions $\varphi = \Phi_1 - \tau_m$, i.e., domain (43). In this case, the pole of f_1 at $\zeta = \zeta_m(\varphi) := -\varphi + [\pi - \tau_m + \Phi_1]$ can cross the saddle point π when φ ranges $[0, \Phi_1]$ (also see Sec. 4.2 in [7] for a similar situation). We consider the disk $B_\pi([\kappa r]^{-1/2+\epsilon})$ centered at $\zeta = \pi$ of a small radius $O([\kappa r]^{-1/2+\epsilon})$. The pole $\zeta_m(\varphi)$ lies in this disk, and we see that the representation ($|\varphi + \tau_m - \Phi_1| \leqslant O(1/(\kappa r)^{1/2+\epsilon})$ holds under the assumption that $-\varphi + \pi - \tau_m + \Phi_1 > \pi$)

$$\begin{aligned} u_1(r,\varphi) = {}& \frac{1}{i\pi}\int_{B_\pi([\kappa r]^{-1/2+\epsilon})\cap\gamma_0^\pi} d\zeta\, f_1(\zeta + \varphi)(\zeta - \zeta_m(\varphi))\frac{e^{\kappa r\cos\zeta}}{\zeta - \zeta_m(\varphi)} - \\ & - 2f_1(-\pi + \varphi)\frac{e^{-\kappa r}}{\sqrt{2\pi\kappa r}}\biggl(1 + O\biggl(\frac{1}{\kappa r}\biggr)\biggr) + \delta u_1(r,\varphi), \end{aligned}$$

where

$$\delta u_1(r,\varphi) = \frac{1}{i\pi}\int_{\gamma_0^\pi \setminus B_\pi([\kappa r]^{-1/2+\epsilon})} d\zeta\, e^{\kappa r\cos\zeta} f_1(\zeta+\varphi),$$

is the remainder that can easily be estimated. We now estimate the integral along an asymptotically small part of the contour γ_0^π in the disk $B_\pi([\kappa r]^{-1/2+\epsilon}) \cap \gamma_0^\pi$. The function $D(\zeta,\varphi) := f_1(\zeta+\varphi)(\zeta-\zeta_m(\varphi))$ is regular in this disk. In the disk, we use the approximation $\cos\zeta = -1 + [\zeta-\pi]^2/2 + \cdots$ and, introducing a new variable $t = i(\zeta-\pi)$, obtain

$$\frac{1}{i\pi}\int_{B_\pi([\kappa r]^{-1/2+\epsilon})\cap\gamma_0^\pi} d\zeta\, D(\zeta,\varphi)\frac{e^{\kappa r\cos\zeta}}{\zeta-\zeta_m(\varphi)} =$$
$$= -\frac{D(\pi,\varphi)e^{-\kappa r}}{i\pi}\int_{-\infty}^{\infty} dt\,\frac{e^{-\kappa r t^2/2}}{t-i[\zeta_m(\varphi)-\pi]}(1+O([\kappa r]^{-1/2+\epsilon})),$$

where we pulled $D(\zeta,\varphi)$ outside the integral by setting $\zeta=\pi$ and replacing the integration limits with $\pm\infty$, which gives an exponentially small relative error. In the last integral, the pole in $i[\zeta_m(\varphi)-\pi]$ is bypassed by the contour from below. In particular, if $\mathrm{Im}(i[\zeta_m(\varphi)-\pi]) \leqslant 0$, then the integration contour bypasses the pole from below along a small-radius arc. The obtained integral is expressed in terms of a Fresnel-type integral in accordance with § 6.3.1 in [29] as

$$\Psi(z;s) := \int_{-\infty}^{\infty} dt\frac{e^{-zt^2}}{t-s} = \pi i e^{-zs^2}[1-\mathcal{F}(-is\sqrt{z}\,)],$$

where

$$\mathcal{F}(\zeta) = \frac{2}{\sqrt{\pi}}\int_0^\zeta e^{-t^2}\,dt.$$

As a result, in a neighborhood of the singular direction, i.e., if $|\varphi+\tau_m-\Phi_1| \leqslant O(1/(\kappa r)^{1/2+\epsilon})$, we obtain

$$u_1(r,\varphi) = -D(\pi,\varphi)\frac{e^{-\kappa r}}{i\pi}\Psi\left(\frac{\kappa r}{2}; i[\Phi_1-\varphi-\tau_m]\right) -$$
$$-2f_1(-\pi+\varphi)\frac{e^{-\kappa r}}{\sqrt{2\pi\kappa r}}\left(1+O\left(\frac{1}{\kappa r}\right)\right) + \delta u_1(r,\varphi), \tag{46}$$

where $\delta u_1(r,\varphi)$ is the contribution of other possibly captured poles. In a neighborhood of a singular direction, the Fresnel-type integral Ψ in (46) plays the role of a transition function that switches the asymptotic regimes of exponential decrease of the eigenfunction in the domain Ω_1. It is worth noting that the asymptotics in the domains Ω_3 and Ω_2 can be studied similarly. In Ω_3, we then consider the poles $\zeta = \pm[\pi-\tau_m+\Phi_1]$ and $\zeta = \pm[\pi-\tau_m+\pi-\Phi_2]$ and calculate the asymptotics; in Ω_2, we consider the "leading" pole at $\zeta = \pm[\pi-\tau_m+\pi-\Phi_2]$. The corresponding calculations are very similar to those described above for the domain Ω_1. We obtain the assertion of Lemma 2.1.

In Fig. 1, the singular directions are shown symbolically. In Ω_1, the angle φ ranges $[0,\Phi_1]$. The number of singular directions is determined by the parameters Φ_1, Φ_2, and γ_1, γ_2 via τ_m. If φ varies and crosses a saddle point, then each pole generates the corresponding singular direction. If φ is small and we consider Ω_1, the asymptotics is determined by the contribution of the saddle points $\pm\pi$ (see (44)), where $H(\varphi+\tau_m-\Phi_1)=0$, whence $u_{1m} \sim e^{-\kappa r}/\sqrt{\kappa r}$. As φ increases, the pole at $\zeta=\zeta_m(\varphi)$ approaches the saddle point π from the right; if $|\varphi+\tau_m-\Phi_1| \leqslant O(1/(\kappa r)^{1/2+\epsilon})$, i.e., near a singular direction (shown in Ω_1, Fig. 1), then the asymptotics is described by expression (46) with a Fresnel-type integral. The pole crosses the saddle point π as φ increases and is captured inside the contour, which leads to asymptotics (44), where $H(\varphi+\tau_m-\Phi_1)=1$, $\varphi+\tau_m-\Phi_1 \geqslant C/(\kappa r)^{1/2+\epsilon}$. The captured pole $\zeta=\zeta_m(\varphi)$ generates a contribution in the form of a decreasing exponential $2A_1^+ e^{-\kappa r\cos(\varphi+\tau_m-\Phi_1)}$, while the contribution of the saddle points is of $O(e^{-\kappa r}/\sqrt{\kappa r})$. Lemma 2.1 implies that the solutions u_m are square-integrable and satisfy estimate (5). Using the Sommerfeld integral representation, we can also show that $u_m(r,\varphi) = C_m + O(r^{\delta_*})$ as $r \to 0$, $\delta_* > 0$, and hence $u_m \in H^1(\omega)$, which completes the construction of the eigenfunction in our example.

Appendix A: Diagonalization of a matrix Mehler operator M. Spectral properties of the scalar Mehler operator

The results in the appendix are based on well-known formulas related to the classical Mehler–Fock transformation, which is underlain by the Mehler formulas of 1881 [27].

By definition, a matrix Mehler operator $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} M & 0 \\ 0 & aM \end{pmatrix},$$

where M is the "scalar" Mehler operator,[5]

$$(M\rho)(x) = \frac{1}{\pi}\int_0^1 \frac{dy\,\rho(y)}{x+y},$$

which is bounded and self-adjoint in $L_2([0,1])$, $0 < a < 1$. Obviously, the spectral properties of $\mathbf{M}$ are determined by the properties of its "scalar" components, $\mathbf{M} = M \oplus (aM)$. The operator M is called the scalar Mehler operator, and its properties are discussed in [28].

Operator M and its resolvent. With the information about the scalar Mehler operator M available, we now consider its matrix analog $\mathbf{M}$. We begin with expression (56) (see below). We write

$$\frac{a}{\pi}\int_0^1 \frac{\mathcal{P}_q(y)}{x+y}\,dy = \frac{a\mathcal{P}_q(x)}{\cosh(\pi q)},$$

and relate the parameter p in (57) and q in the last equation by the formula

$$\cosh(\pi q) = a\cosh(\pi p), \qquad a = \frac{\gamma_2}{\gamma_1} \leqslant 1. \tag{47}$$

Transcendental equation (47) has the solution

$$q(p) := \frac{1}{\pi}\operatorname{arcosh}(a\cosh(\pi p)),$$

whose branch is determined as follows. We consider the set of cuts $b_* + im$, $m = 0, \pm1, \pm2, \dots$, where $b_* = [-a_*/\pi, a_*/\pi]$ ($a_* = \operatorname{arcosh}(a^{-1})$) on the complex plane p. We introduce a holomorphic function $q(p)$ defined in (47), which takes the domain outside this periodic system of cuts to the complex plane of the variable q with a periodic system of cuts along $b^* + im$, $m = 0, \pm1, \pm2, \dots$, where $b^* = [-ia^*/\pi, ia^*/\pi]$ ($a^* = \arccos(a)$). The map $q(\,\cdot\,)$ has the following properties:

- $q(p) = -q(-p)$;
- $q(p+im) = q(p) + im$;
- $q(p) = p + \frac{1}{\pi}\ln a + O(p^{-1})$, $p \to \infty$.

[5] A similar operator was studied in [27] in different terms. The operator M was considered in connection with the Dixon integral equation in monograph [30].

We then find a (generalized) eigenfunction of the operator $\mathbf{M}$

$$\mathbf{P}_p(x) = \begin{pmatrix} \mathcal{P}_p(x) \\ \mathcal{P}_{q(p)}(x), \end{pmatrix}, \tag{48}$$

corresponding to $\mu(p) = 1/\cosh(\pi p)$:

$$\mathbf{M}\mathbf{P}_p(x) = \mu(p)\mathbf{P}_p(x). \tag{49}$$

When the parameter p ranges all nonnegative values, the spectral parameter $\mu(p) = 1/\cosh(\pi p)$ takes all values on the interval $[0,1]$. The essential (absolutely continuous) spectrum $\sigma_{\mathrm{e}}(\mathbf{M}) = [0,1]$ of $\mathbf{M}$ has multiplicity 2 because the operator can be represented by the orthogonal sum $M \oplus (aM)$. Obviously, eigenvector (48) can be represented by an orthogonal sum of vectors of the form $(0, \mathcal{P}_{q(p)}(x))^{\mathrm{T}}$ and $(\mathcal{P}_p(x), 0)^{\mathrm{T}}$.

The formula for the resolvent of $\mathbf{M}$ follows directly from the results in [28],

$$\mathbf{u}(x) = [\mathbf{M} - \mu\mathbf{I}]^{-1}\mathbf{f}(x) = -\frac{1}{\mu}\{\mathbf{I} + \mathbf{A}_\mu\}\mathbf{f}(x), \tag{50}$$

where

$$\mathbf{A}_\mu \mathbf{f}(x) = \frac{1}{\pi}\int_0^1 \mathbf{a}(x,y;\mu)\mathbf{f}(y)\,dy,$$
$$\mathbf{a}(x,y;\mu) = \begin{pmatrix} a_\mu(x,y) & 0 \\ 0 & a_{\mu/a}(x,y) \end{pmatrix},$$
$$a_{\mu/a}(x,y) = \pi\int_0^\infty \frac{\mathcal{P}_q(x)\mathcal{P}_q(y)}{a^{-1}\mu\cosh(\pi q) - 1}\,dq.$$

Using an estimate from [28], we obtain

$$\|\mathbf{a}(x,y,\mu)\|_{\mathbb{C}^2\to\mathbb{C}^2} \le C\frac{|\ln(2/x)\ln(2/y)|}{\sqrt{xy}} \tag{51}$$

uniformly in μ assuming that $\mu \in \mathcal{B}_1$, i.e., near the end of the spectrum $\mu = 1$.

The partition of unity $\mathbf{E}_t$ has the form

$$\begin{aligned} &\mathbf{E}_t\mathbf{g}(x) = 0, \qquad t \leqslant 0, \\ &\mathbf{E}_t\mathbf{g}(x) = \mathbf{g}(x) + \frac{1}{\pi}\int_0^1 \mathbf{e}(x,y;t)\mathbf{g}(y)\,dy, \qquad t \in (0,1], \\ &\mathbf{E}_t\mathbf{g}(x) = \mathbf{g}(x), \qquad t > 1, \end{aligned} \tag{52}$$

where

$$\mathbf{e}(x,y;t) = \begin{pmatrix} e(x,y;t) & 0 \\ 0 & e_a(x,y;t) \end{pmatrix}, \tag{53}$$

$e_a(x,y;\mu) = e(x,y;\mu/a)$ for $\mu/a \in (0,1]$, and $e_a(x,y;\mu) = 0$ for $\mu/a > 1$. The kernel $e(x,y;\mu)$ has the form

$$\begin{aligned} &e(x,y;\mu) = -\int_0^1 \frac{d\tau}{\tau}\frac{H(\tau-\mu)}{\sqrt{1-\tau^2}}\mathcal{P}_{p(\tau)}(x)\mathcal{P}_{p(\tau)}(y), \qquad \mu \in (0,1), \\ &p(\tau) = \frac{1}{\pi}\ln\left(\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 1}\right), \qquad p(\tau) \ge 0, \qquad p(\tau) \to \infty \quad \text{as} \quad \tau \to 0+. \end{aligned} \tag{54}$$

With the necessary information about the unperturbed operator $\mathbf{M}$ available, we use traditional methods to describe its perturbation by a compact self-adjoint operator $\mathbf{V}$. In our case, a relatively smooth perturbation $\mathbf{V}$ is an integral operator.

Using these results and some additional considerations, it is easy to diagonalize the scalar Mehler operator M. It is known that M has a simple absolutely continuous spectrum $\sigma_a(M) = [0,1]$ and the continuous spectrum eigenfunctions have been determined explicitly (see [28]):

$$\mathcal{P}_p(x) := \frac{\sqrt{p\tanh(\pi p)}}{x} P_{ip-1/2}\left(\frac{1}{x}\right)$$

with the asymptotics (see [31], 8.772(1))

$$\mathcal{P}_p(x) = \frac{\sqrt{p\tanh(\pi p)}}{x}\left(\frac{\Gamma(-ip)}{\Gamma(-ip+1/2)}\left[\frac{x}{2}\right]^{1/2-ip} + \right.$$
$$\left. + \frac{\Gamma(ip)}{\Gamma(ip+1/2)}\left[\frac{x}{2}\right]^{1/2+ip}\right)\left(\frac{1}{\sqrt{\pi}} + O(x^2)\right),$$

$x \to 0+$, $p > 0$, and $\mathcal{P}_p(x) = O(1)$ for $p \to \infty$, $1 \geqslant x > 0$. The functions $\mathcal{P}_p(x)$ are real for $p \geqslant 0$, in particular, $\mathcal{P}_0(x) > 0$. We use the following assertion [28].

Theorem A.1. *The modified Mehler–Fock transformation defined by the formulas*

$$F(x) = \int_0^\infty \mathcal{P}_p(x) F^*(p)\, dp, \tag{55}$$

$$F^*(p) = \int_0^1 \mathcal{P}_p(x) F(x)\, dx, \tag{56}$$

is unitary, $\mathcal{U}\colon L_2(0,\infty) \to L_2(0,1)$. *The Mehler–Fock transformation diagonalizes the Mehler operator* M,

$$\frac{1}{\pi}\int_0^1 \frac{\mathcal{P}_p(y)}{x+y}\, dy = \frac{\mathcal{P}_p(x)}{\cosh(\pi p)}. \tag{57}$$

Appendix B: Maliuzhinets functional equations for f_1, f_2, and f_3

The integral representations for solutions (41) of the eigenfunction problem satisfy the equation $-\triangle u = Eu$ in Ω and can be substituted in the boundary conditions. In this way, we obtain functional equations called the Maliuzhinets equations for the Sommerfeld transforms f_1, f_2, and f_3. In particular, they allow continueing f_1, f_2, and f_3 from the regularity strips described in Lemma 6.1 to the whole complex plane. Moreover, it turns out that all poles are located on the real axis. From the Neumann condition on the half-lines $\varphi = 0$ and $\varphi = \pi$, we necessarily obtain

$$f_1(\zeta) = -f_1(-\zeta), \qquad f_3(\zeta) = -f_3(-\zeta), \tag{58}$$

which has already been established above. The continuity condition for l_1 implies (see for details of calculations in a similar situation in [7])

$$f_1(\zeta + \Phi_1) - f_1(-\zeta + \Phi_1) = f_2(\zeta + \Phi_1) - f_2(-\zeta + \Phi_1). \tag{59}$$

We now consider the second condition on l_1,

$$\frac{1}{\kappa r}\left(\frac{\partial u_1}{\partial\varphi}-\frac{\partial u_2}{\partial\varphi}\right)\Big|_{\varphi=\Phi_1}-\frac{\gamma_1}{\kappa}u_1(r,\Phi_1)=$$
$$=\frac{1}{2i\pi}\int_{\gamma_0}d\zeta\,\frac{e^{\kappa r\cos\zeta}}{\kappa r}(f_1'(\zeta+\Phi_1)-f_1'(-\zeta+\Phi_1)-[f_1'(\zeta+\Phi_1)-f_1'(-\zeta+\Phi_1)]-$$
$$-\frac{\gamma_1}{\kappa}[f_1(\zeta+\Phi_1)-f_1(-\zeta+\Phi_1)])=0.$$

Integrating by parts, we obtain

$$\frac{1}{2i\pi}\int_{\gamma_0}d\zeta\,e^{\kappa r\cos\zeta}(\sin\zeta(f_1(\zeta+\Phi_1)+f_1(-\zeta+\Phi_1)-[f_1(\zeta+\Phi_1)+f_1(-\zeta+\Phi_1)])-$$
$$-2\Lambda[f_1(\zeta+\Phi_1)-f_1(-\zeta+\Phi_1)])=0.$$

By the theorem on the inversion of the Sommerfeld integral (see [17], Sec. 3.4), Eq. (59) gives

$$(\sin\zeta-\sin\tau_m)f_1(\zeta+\Phi_1)-(-\sin\zeta-\sin\tau_m)f_1(-\zeta+\Phi_1)=$$
$$=(\sin\zeta+\sin\tau_m)f_2(\zeta+\Phi_1)-(-\sin\zeta+\sin\tau_m)f_2(-\zeta+\Phi_1). \tag{60}$$

Similarly, from the boundary condition for l_2, we obtain

$$f_2(\zeta+\Phi_2)-f_2(-\zeta+\Phi_2)=f_3(\zeta+\pi-\Phi_2)-f_3(-\zeta+\pi-\Phi_2) \tag{61}$$

and

$$(\sin\zeta-a\sin\tau_m)f_2(\zeta+\Phi_2)-(-\sin\zeta-a\sin\tau_m)f_2(-\zeta+\Phi_2)=$$
$$=(\sin\zeta+a\sin\tau_m)f_3(\zeta+\pi-\Phi_2)-(-\sin\zeta+a\sin\tau_m)f_3(-\zeta+\pi-\Phi_2). \tag{62}$$

We recall that $a=\gamma_2/\gamma_1$, $a\sin\tau_m=:\sin t_m$. Maliuzhinets functional equations (58)–(62) allow extending the transforms to the whole complex plane as meromorphic functions. They also allow asserting that all poles are located on the real axis. This follows from the fact that translations in the arguments in the equations are directed along the real axis, while the poles of f_1, f_3 that are closest to the imaginary axis are real and located symmetrically at the respective points $\zeta=\pm[\pi-\tau_m+\Phi_1]$, $\zeta=\pm[\pi-\tau_m+\pi-\Phi_2]$. This is also true for $f_2(z)$ with respect to the line $\frac{1}{2}(\Phi_1+\Phi_2)+i\mathbb{R}$ for the poles at $\zeta=-[\pi-\tau_m]+\Phi_1$ and $\zeta=[\pi-\tau_m]+\Phi_2$. We note that the transforms f_1, f_2, f_3 are bounded at $\pm i\infty+\mathrm{Re}(\zeta)$ for fixed $\mathrm{Re}(\zeta)$.[6] Now the problem for f_1, f_2, and f_3 satisfying Maliuzhinets equations (58)–(62) amounts to finding $\Lambda_m=\sin\tau_m$ and nontrivial solutions that are regular in the strips in Lemma 6.1 and bounded at infinity. In the correct interpretation, this problem is equivalent to the spectral problem for FD equations (11). If we have appropriate nontrivial solutions, then we can reconstruct eigenfunctions by using Sommerfeld integrals (41). But we have already verified the existence of $\Lambda=\sin\tau_m$ and the corresponding H_{1m}, H_m, H_{3m}, and $\tilde{h}_m$ determining the transforms f_1, f_2, and f_3 in (36), (38), and (40) given by meromorphic functions with real poles.

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[6]Boundedness is ensured by a proper behavior of u as $r\to 0$.

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